ON CRITICAL GRAPHS, INDECOMPOSABLE GRAPHS AND PERFECT GRAPHS

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To

MY PARENTS

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CERTIFICATE

This is to certify that the Ph.D. thesis entitled "On Critical Graphs, Indecomposable Graphs and Perfect Graphs" by Mr. George O.T. is a record of bonafide research work carried out by him under my supervision and guidance. He had fulfilled the other requirements for the award of Ph.D. degree. The results embodied in this thesis have not been submitted to any other Institute or University for the award of any degree or diploma.

I.I.T., Kanpur
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List of Publications

1. B-graphs, J. Math. Phy. Sci (1980).
2. On B-graphs, Indian J. Pure and Applied Maths (1981).
3. On the strong perfect graph conjecture and critical graphs, Disc. Maths. (1981).

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INTRODUCTION

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1.1 General Introduction

Interest in the study of colouring the vertices of a graph came because of the celebrated Four Colour Conjecture (4.C.C.). This conjecture deals with the question of colouring the points of a planar graph with four or fewer colours so that adjacent points have different colours. As a result, 4.C.C. acted as a catalyst for getting a good number of results in the field of vertex colouring of graphs. Heawood [61] established that every planar graph is 5-colourable. A characterization of bicolourable graphs was given by König [67]. The problem of determining the chromatic number $\chi(G)$ of an arbitrary graph seems to be extremely difficult. Many bounds are known for $\chi(G)$ in terms of invariants of graphs. It goes without saying that the number of points in a largest complete subgraph of a graph G is always a lower bound for $\chi(G)$. Szekeres and Wilf [116] supplied the first upper bound of the chromatic number of a graph depending on the maximum degree of the graph. From Brooks [25], one can observe that this bound may be improved in certain cases. Do graphs with large chromatic number possess large cliques? This question was asked by Dirac [33].

The answer to this is not in the affirmative. This fact was established independently by Descartes [30], Mycielski [80], Zykov [134] and Kelly and Kelly [66]. Shannon [108] observed that graphs G for which $\alpha(G) = \Theta(G)$ represent perfect channels in communication theory. The α-perfect property for triangulated graphs was established by Hajnal and Suranyi [53]. Two years later Berge [9] proved that they satisfy the χ-perfect property also. The concept of a perfect graph was introduced by Berge [10]. He conjectured that a graph is perfect if and only if its complement is perfect. This has become known as the perfect graph conjecture. The class of triangulated graphs was the first batch of graphs to be recognized as being perfect. However, this property does not hold good for infinite graphs. We refer to Hajnal and Suranyi [53], Perles [94], Nash-Williams [81], Baumgartner, Malitz and Reinhardt [3], Trotter [120], and Wagon Stanley [128] for the details.

Soon after the introduction of perfect graphs, many started to identify these graphs. Berge [13] showed that many familiar classes of graphs such as triangulated graphs, comparability graphs, bipartite graphs, interval graphs, unimodular graphs and line graphs of bipartite graphs are perfect. In the year 1962 Berge [11] gave one more conjecture on perfect graphs. It states that for a graph G the following conditions are equivalent.

1) G is α-perfect

2) G is χ-perfect

3) G and $\bar{G}$ do not contain an induced subgraph isomorphic to C_{2k+1}, $k \geq 2$.

This conjecture is known as the strong perfect graph conjecture (SPGC). An odd cycle of length ≥ 5 is an imperfect graph. From the definition of a perfect graph one can see that if a graph G contains such a cycle as an induced subgraph, then G is imperfect. In [44] Gallai proves that if G is a graph with the property that every odd cycle of length ≥ 5 has two noncrossing chords, then G is perfect. Suranyi [112] gave a shorter proof of this result. The case when each odd cycle has two crossing chords was proved by Olaru [83]. Meyniel [78] establishes that if G is a graph such that every odd cycle of length ≥ 5 has at least two chords, then G is perfect.

Interval graphs (a class of perfect graphs) became popular because of their applications in traffic light phasing, seriation (that is an attempt to place a set of items in their proper chronological order) and certain other optimization problems. Hajos [54] first posed the problem of characterizing interval graphs. The famed molecular biologist Benzer [8] was also trying to find the answer to a related problem in his investigations of the fine structure of the gene. The first characterization of interval graphs appeared in 1962 by Lekkerkerker and Boland [70] followed by Gilmore and Hoffman [46]. Fulkerson

and Gross [42] gave still another characterization. Interval graphs are perfect because they are triangulated. Closely related to interval graphs are circular-arc graphs. Every interval graph is a circular-arc graph. But the converse is not true. Circular-arc graphs are not perfect in general. For example C_{2k+1}, $k \geq 2$ belongs to this class and is not perfect. We refer to [109], [111], [74], [64] and [126] for applications of these graphs.

Permutation graphs and split graphs are two other classes of perfect graphs. Pnueli, Lempel and Even [97] proved that a graph G is a permutation graph if and only if G and $\bar{G}$ are comparability graphs. This implies that permutation graphs are perfect. Földes and Hammer [37] established that split graphs belong to the class of triangulated graphs. Thus these graphs also are perfect. The idea of splittance of a graph namely the minimum number of edges to be added or deleted in order to produce a split graph was introduced by Hammer and Simeone [55]. According to this definition, split graphs are those graphs whose splittance is zero.

During the initial stages the properties of α-perfectness and χ-perfectness were thought to be distinct. All known α-perfect graphs were found to be χ-perfect. This might have prompted Berge [10] to conjecture that a graph is α-perfect if and only if it is χ-perfect. Lovasz ([71], [72]) settled this conjecture in the affirmative. Now in order to establish

that a graph is perfect we need prove only either α-perfectness or χ-perfectness. Fulkerson ([38], [39], [40], [41]) almost proved this conjecture. Only he did not get the credit for it. But he could console himself that in the process of his investigations he could bring about the concept of antiblocking pairs of polyhedra which has become an asset in the field of polyhedral combinatorics. Olaru [86] also gave a proof of the equivalence of α-perfectness and χ-perfectness of graphs.

After Lovasz' characterization [72] of perfect graphs the SPGC is simplified. It asserts that a graph G is perfect if and only if it does not contain C_{2k+1} or $\overline{C_{2k+1}}$, $k \geq 2$ as an induced subgraph of G. Many graph theorists were under the impression that a good characterization of critical graphs would settle the Berge's conjecture. Keeping this in view, they started to find many structural properties of critical graphs. House [63], Mel'nikov and Vizing [77], Lovasz [73], Olaru [85], Markosjan [75], Weinstein [129], Chvatal [27], Tucker [125], Greenwell Don [51], Bland, Huang and Trotter [21], Broere and Mynhardt [24] are some of them. Some graph theorists tried to prove the validity of the SPGC for some particular classes of graphs. Tucker ([122], [124]), proved that this conjecture is true for planar graphs and circular-arc graphs. Parthasarathy and Ravindra ([92], [93]) established the conjecture for $K_{1,3}$-free graphs and (K_4-e)-free graphs. The fact that webs satisfy the SPGC was proved by Trotter [118].

Matrices and polyhedra were also used in the study of perfect and critical graphs. Hedetniemi [62], Commoner [29], Padberg ([89], [90], [91]) and Tamir [117] discussed conditions for the perfectness of graphs in terms of matrices. Chvatal [26] gave a polyhedral characterization of perfect graphs. Padberg ([88], [89]) supplied polyhedral characterization of critical graphs. He obtained some important properties of critical graphs with the help of this polyhedral approach. Edmonds [35], Monma and Trotter [79] also deal with perfect graphs and polyhedra with (0,1)-valued extreme points. Rao and Ravindra [99] gave still another characterization of perfect graphs. Nordhaus [82], Olaru ([84], [87]), Korner [68], Tucker [123], Seinsche [107], Arditti and de Werra [2], Karpetjan [65], Markosjan and Karpetjan [76], Roberts [105], Ravindra [102], Ravindra and Parthasarathy [104], Bollobas [22], Trotter [119] and Golumbic ([48], [49], [50]) have obtained some more results on perfect graphs. For some interesting other references on perfect graphs and the SPGC, we refer to Berge ([12], [14], [16], [17], [18], [19], [20]), Tucker ([121], [127]), Ravindra [100], de Werra [31], Grinstead [52], Wessel Walter [133], Chvatal, Graham, Perold and Whitesides [28].

The critical (edge colouring) graph conjecture states that every critical graph has an odd number of vertices. The only critical graphs observed to date are the odd cycles C_{2k+1}, $k \geq 2$ and their complements. Hence this conjecture is applicable to

critical graphs also. So it is quite natural to expect that the solution of any one of them would give a clue to the solution of the other. Gold'berg [47] produced a critical (edge colouring) graph with an even number of vertices. Unfortunately this counter example is not a critical graph.

Point core, line core, indecomposable graphs and line critical graphs are some concepts related to the independence number of a graph. Beineke, Harary and Plummer [7], Plummer [95], Wessel Walter ([130], [131], [132]), Berge [15], Krieger Michael [69], Suranyi ([113], [114], [115]) discussed critical lines of a graph and line critical graphs. We refer to Harary and Thomassen [60] for an interesting reading on critical graphs. Dulmage and Mendelsohn [34] defined the core of a bipartite graph. Harary and Plummer ([57], [58]) extended this idea to a general graph. Analogous to this they introduced the point core of a graph. Rao [98] developed tight bounds for the sum and product of the line core numbers as well as the point core numbers of complementary graphs. Dirac [32] and Andrasfai [1] discussed graphs critical with respect to independence number. The idea of an indecomposable graph was introduced by Erdős and Gallai [36]. Harary and Plummer [59] made a detailed study of these graphs. Apart from proving some properties of these graphs they established that line critical graphs formed a subset of the set of indecomposable graphs.

The recent strategy of graph theorists towards the solution of the SPGC is to identify certain classes of graphs which possess one particular fundamental property of critical graphs so that critical graphs will be a subclass of the new class of graphs. Any characterization of this new class of graphs will lead to a characterization of critical graphs as well. This idea gave birth to partitionable graphs ([89],[27]) well-covered graphs ([96], [101], [110]) and B-graphs [103].

A brief summary of the important results presented in the thesis is given below.

In Chapter 2 we prove that for a critical graph $\alpha < \theta$ and $\omega < \chi$; $\theta = \alpha+1$; $\chi = \omega+1$; $\chi(H_1(v)) = \chi-2$; $\theta(H_2(v)) = \theta-2$; $2(p-1)^{\frac{1}{2}} \leq \omega + \alpha \leq \frac{p+3}{2}$; $2(p-3) \leq |H_1| \cdot |H_2| \leq (\frac{p-1}{2})^2$. It is established that the minimum number of vertices of a critical graph of even order, if it exists, is twentysix. Among all graphs up to sixteen vertices C_{2n+1} and $\overline{C_{2n+1}}$, $2 \leq n \leq 7$ are the only critical graphs. It is proved that a critical and $\overline{K_{1,3}}$ -free graph G has $d(G,v) = p+1 - 2\alpha$. The validity of the SPGC is established for $\overline{K_{1,3}}$ -free graphs.

In Chapter 3 some of the graphs proved to be B-graphs include P_{2n}, C_n, an acyclic graph without isolated vertices and having a perfect matching, powers of cycles, line graphs of powers of cycles and join of B-graphs with the same independence number. Necessary and sufficient condition for the join of two

B-graphs to be a B-graph and that for an acyclic graph to be a B-graph are given. An algorithm is found to generate new B-graphs from known ones. P_{2n+1}, C_n, powers of cycles, line graphs of powers of cycles are proved to be B*-graphs. Necessary and sufficient condition for a path to be a B*-graph is given. Cycles, powers of cycles and line graphs of powers of cycles are proved to belong to the class of B**-graphs. It is established that an acyclic graph cannot be a B**-graph.

In Chapter 4 among the B-point critical graphs studied are C_{2n}, C_{2r}^{r-1}, C_{nr}^{n-1} and $G_1 + G_2$, where G_1 and G_2 are B-point critical with $\alpha(G_1) = \alpha(G_2)$. An algorithm is given for generating more B-point critical graphs from known ones. A counter example is given to disprove a conjecture, namely a graph G is B-point critical implies that $\theta(G) = \alpha(G)$. Various generalizations of this counter example are discussed.

In Chapter 5 indecomposable graphs other than complete graphs are proved to be imperfect. Critical graphs are found to be indecomposable. A graph G with $\alpha = 2$ and $\theta \geq 3$ is shown to be indecomposable. A construction is supplied in order to generate indecomposable graphs. Conditions are given for C_{nr+k}^{n-1} and $\overline{C_{nr+k}^{n-1}}$ to be indecomposable. Line graphs of powers of odd cycles are proved to belong to the class of line critical graphs. Of the two classes of graphs $T(K_{2n})$ and $T(K_{2n+1})$, the former is found to be line critical and the latter decomposable. The fact that the stability number of an

indecomposable graph is unaffected by the removal of any complete subgraph of the graph is proved.

In Chapter 6 the existence of an odd induced cycle of length ≥ 5 or its complement in C_m^k ($m = 2n$ or $2n+1$), for $2 \leq k \leq n-2$ is established. The perfectness of the remaining powers of cycles is discussed. Using this, it is proved that the SPGC holds good for powers of cycles. Four conjectures are given. It is established that each of these conjectures is equivalent to the SPGC.

Seven papers have been communicated based on these results out of which three papers have been accepted for publication. They are listed as follows :

1. B-graphs, J. Math. Phy. Sci (1980)
2. On B-graphs, Indian J. Pure and Applied Maths (1981)
3. On the strong perfect graph conjecture and critical graphs, Disc. Maths (1981).

1.2 Definitions

In this section D stands for definition.

D1.2.1 A graph G consists of a finite nonempty set V of points (or vertices) together with a prescribed set E of unordered pairs of distinct points of V. We may choose to mention vertices or points freely in this thesis.

D1.2.2 Each pair $e = (u,v)$ or uv of points in E is called a line (or edge) of G and e is said to join u and v or incident to u and v and the points u and v are adjacent.

D1.2.3 If two distinct lines e_1 and e_2 are incident with a common point, then they are adjacent lines.

D1.2.4 A graph G is called a labelled graph when the points are distinguished from one another by labels such as $v_1, v_2, v_3, \ldots, v_p$ where $|V| = p$.

D1.2.5 Two graphs G and H are isomorphic if there exists a one-to-one correspondence between their point sets which preserves adjacency. If G and H are isomorphic we write $G = H$.

D1.2.6 A subgraph of G is a graph having all of its points and lines in G. If G_1 is a subgraph of G, then G is a supergraph of G_1.

D1.2.7 For any set S of points of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with point set S. Thus two points of S are adjacent in $\langle S \rangle$ if and only if they are adjacent in G.

D1.2.8 A spanning subgraph is a subgraph containing all the points of G.

D1.2.9 The removal of a point v_i from a graph G results in the subgraph $G-v_i$ of G consisting of all points of G except v_i

and all lines not incident with v_i. Thus $G-v_i$ is the maximal subgraph of G not containing v_i.

D1.2.10 The removal of a line e_j from G yields the spanning subgraph $G-e_j$ containing all lines of G except e_j. Thus $G-e_j$ is the maximal subgraph of G not containing e_j. The removal of a set of points or lines from G is defined by the removal of these elements in succession.

D1.2.11 A walk of a graph G is an alternating sequence of points and lines v_o, e_1, v_1, e_2, v_2, ..., v_{n-1}, e_n, v_n beginning and ending with points, in which each line is incident with the two points immediately preceding and following it. This walk joins v_o and v_n and may also be denoted by $v_o\ v_1\ v_2\ \ldots\ v_n$ (the lines being evident by context). It is closed if $v_o = v_n$.

D1.2.12 A walk is a path if all the points (and thus all the lines) are distinct.

D1.2.13 A closed path is called a cycle.

D1.2.14 We denote by C_n the graph consisting of a cycle with n points and by P_n a path with n points; C_3 is often called a triangle.

D1.2.15 Suppose that G is a graph consisting of a cycle C_n together with an edge which joins two nonadjacent vertices of the cycle. Then we say that G is a cycle with a chord.

D1.2.16 A graph is connected if every pair of points is joined by a path. A maximal connected subgraph of G is called a connected component or simply a component of G. A graph which is not connected is a disconnected graph.

D1.2.17 The length of a walk, path or cycle is the number of lines in it.

D1.2.18 The distance $d(u,v)$ between two points u and v in a connected graph G is the length of a shortest path joining them.

D1.2.19 The k^{th} power denoted by G^k of a graph G has $V(G^k) = V(G)$ with u,v adjacent in G^k whenever $d(u,v) \leq k$ in G.

D1.2.20 The degree of a point v in a graph G denoted by $d(G,v)$ is the number of lines incident with v. If there is no confusion about the graph, we may represent it by $d(v)$ also.

D1.2.21 The complement $\overline{G}$ of a graph G has $V(G)$ as its point set. Two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A self complementary graph is isomorphic with its complement.

D1.2.22 $[x]$ denotes the greatest integer not exceeding the real number x and $\{x\}$, the smallest integer not less than x.

D1.2.23 A bigraph (or bipartite graph) G is a graph whose point set V can be partitioned into two subsets V_1 and V_2 such

that every line of G joins a vertex of V_1 with a vertex of V_2. If each vertex of V_1 is adjacent to all the vertices of V_2, then G is a complete bipartite graph.

D1.2.24 A clique of a graph is a maximal complete subgraph.

D1.2.25 The complete graph K_p has every pair of its p points adjacent.

D1.2.26 If all the vertices of a graph have the same degree k then it is called a k-regular graph.

D1.2.27 A graph G is called an interval graph if its vertices can be put into one-to-one correspondence with a set of intervals on the real line such that two vertices are connected by an edge of G if and only if their corresponding intervals have nonempty intersection.

D1.2.28 In a triangulated graph every cycle of length strictly greater than three has a chord.

D1.2.29 The number of vertices of a graph G is called the order of G.

D1.2.30 A clique of a graph G is maximum if there is no clique of G of larger order. The number of vertices in a maximum clique of G is called the clique number (density) of G denoted by $\omega(G)$ or by simply ω .

D1.2.31 A subset S of vertices of a graph G is a stable set (or independent set) if no two vertices of S are adjacent. The number of vertices in a maximum stable set is called the stability number (or independence number) of G denoted by $\alpha(G)$ or α. A maximum independent set is denoted by MISS.

D1.2.32 A clique cover of order k is a partition of the vertices $V = V_1 \cup V_2 \cup \ldots \cup V_k$ such that each $\langle V_i \rangle$ is a clique. The smallest possible clique cover of G is called the clique cover number (or covering number) of G represented by $\theta(G)$ or θ.

D1.2.33 A colouring of a graph is an assignment of colours to its points so that no two adjacent points have the same colour. The set of all points with any one colour is independent and is called a colour class. The minimum number of colours needed to colour all the vertices of a graph G subject to the above condition is the point chromatic number $\chi(G)$ of G. We prefer to call it simply the chromatic number. If there is no misunderstanding about the graph involved we may represent it by χ also.

D1.2.34 Let V_1, V_2 and E_1, E_2 be the point sets and line sets of G_1 and G_2 respectively. The union of G_1 and G_2, denoted by $G = G_1 \cup G_2$ has $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The join of G_1 and G_2 is denoted by G_1+G_2 and consists of $G_1 \cup G_2$ and all line joining V_1 with V_2.

D1.2.35 A graph is acyclic if it has no cycles.

D1.2.36 The line graph $L(G)$ of G is defined to be the graph whose vertices correspond to the edges of G; and two vertices of $L(G)$ are joined by an edge if and only if they correspond to adjacent edges in G.

D1.2.37 The points and lines of a graph are called its elements. Two elements of a graph are neighbours if they are either incident or adjacent. The total graph $T(G)$ has point set $V(G) \cup E(G)$ and two points of $T(G)$ are adjacent whenever they are neighbours in G.

D1.2.38 A subset M of E is called a matching in G if no two elements of M are adjacent in G. The two adjacent points of an edge in M are said to be matched under M. A matching M saturates a vertex v, and v is said to be M-saturated, if some edge of M is incident with v; otherwise v is M-unsaturated.

D1.2.39 If every vertex of G is M-saturated, the matching M is perfect.

D1.2.40 M is a maximum matching if G has no matching M' with $|M'| > |M|$.

D1.2.41 A point and a line are said to cover each other if they are incident. A set of points which covers all the lines of a graph G is called a point cover for G. The smallest number of points in any point cover for G is called its point covering number denoted by $\alpha_o(G)$ or α_o.

D1.2.42 A set of lines which covers all the points is a line cover. This set spans the graph. The smallest number of lines in any line cover of G is called its line covering number $\beta_o(G)$ or β_o. Refer to D1.2.39 also.

D1.2.43 A set of lines in G is independent if no two of them are adjacent. The largest number of lines in such a set is called the line independence number $\beta(G)$ or β.

D1.2.44 If $\alpha_o(G-v) < \alpha_o(G)$, then v is called a critical point.

D1.2.45 If $\alpha_o(G-e) < \alpha_o(G)$, then e is a critical edge of G.

D1.2.46 A graph in which every point is critical is called point critical.

D1.2.47 A graph in which every line is critical is line critical.

D1.2.48 A graph G is α-perfect if $\alpha(H) = \Theta(H)$ for every induced subgraph H of G.

D1.2.49 A graph G is χ-perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

D1.2.50 A graph is perfect if it is α-perfect (or equivalently χ-perfect). A graph which is not perfect is called imperfect.

D1.2.51 G is said to be critical if it is not perfect and G-v is perfect for every $v \in V$.

D1.2.52 A graph G is B-graph if every vertex of G is in a maximum independent set of G.

D1.2.53 G is B-point critical if G is a B-graph and G-v for every $v \in V$ is not a B-graph.

D1.2.54 A graph G is a B*-graph if every edge of G is in a maximum independent set of edges.

D1.2.55 G is a B**-graph if G is a B-graph as well as a B*-graph.

D1.2.56 Let $[V_1, V_2, \ldots, V_r]$, $r > 1$ be a partition of V(G), the set of points of G. Let G_i be the subgraph of G spanned by V_i, for $i = 1, 2, \ldots, r$. If $\sum_{i=1}^{r} \alpha_0(G_i) = \alpha_0(G)$, then we say that G is decomposable and that $[G_1, G_2, \ldots, G_r]$ is a decomposition of G. A graph which is not decomposable is called indecomposable.

D1.2.57 If H is a graph on fewer points than G, G is said to be H-free if G has no induced subgraph isomorphic to H.

D1.2.58 For $v \in V$ let $N(G,v) = \{u \in V : (u,v) \in E\}$, $\overline{N}(G,v) = \{u \in V : (u,v) \notin E\}$. We represent the induced subgraph of G on N(G,v) and $\overline{N}(G,v)$ by $H_1(G,v)$ and $H_2(G,v)$ respectively. We may also represent them by N(v), $\overline{N}(v)$, $H_1(v)$ and $H_2(v)$. For $H_1(v)$ and $H_2(v)$ we may use H_1 and H_2.

D1.2.59 A graph is said to be a comparability graph if it is possible to give orientation to each edge in such a way that

the relation there is an oriented edge going from vertex a to vertex b, or in short, $a > b$, is a strict order. That is 1) $a > b$, $b > c$ implies $a > c$. 2) $a > b$ implies that not $b > a$.

D1.2.60 Suppose π is a permutation of the numbers 1,2, ...,n. Let us consider π as the sequence $[\pi_1, \pi_2, \ldots, \pi_n]$. π_i^{-1} is the position in the sequence where the number i can be found. We can construct an undirected graph $G[\pi]$ from π in the following manner. $G[\pi]$ has vertices numbered from 1 to n; two vertices are joined by an edge if the larger of their corresponding numbers is to the left of the smaller in π (that is, they occur out of their proper order reading left to right). In other words if π is a permutation of the numbers 1,2, ...,n then the graph $G[\pi] = (V,E)$ is defined as follows :

$V = \{1,2, \ldots, n\}$ and $(i,j) \in E \Leftrightarrow (i-j)(\pi_i^{-1} - \pi_j^{-1}) < 0$.

A graph G is called a permutation graph if there exists a permutation π such that $G = G[\pi]$.

D1.2.61 A graph $G = (V,E)$ is defined to be split if there is a partition $V = S \dot{\cup} K$ of its vertex set into a stable set S and a clique K.

D1.2.62 Let $\mathcal{F}$ be a family of nonempty sets. The intersection graph of $\mathcal{F}$ is obtained by representing each set in $\mathcal{F}$ by a vertex and connecting two vertices by an edge if and only if their corresponding sets intersect.

D1.2.63 The intersection graphs obtained from collection of arcs on a circle are called circular-arc graphs.

D1.2.64 For integers $n \geq 2$ and k, $1 \leq k \leq \frac{n}{2}$ the web $W(n,k)$ has vertices $V_n = \{1,2, \ldots, n\}$ and edges $\{(i,j);\ j = i + k, \ldots i + n - k,$ for $i \in V_n$ (sums modulo n)$\}$.

D1.2.65 A hole is C_{2n+1}, $n \geq 2$ and an antihole $\overline{C_{2n+1}}$ is the complement of a hole.

D1.2.66 The Four Colour Conjecture states that any map on a plane or the surface of a sphere can be coloured with only four colours so that no two adjacent countries have the same colour. Each country must consist of a single connected region, and adjacent countries are those having a boundary line (not merely a single point) in common.

D1.2.67 A graph is said to be embedded in a surface S when it is drawn on S so that no two edges intersect. A graph is planar if it can be embedded in the plane.

D1.2.68 The chromatic index (or edge chromatic number) $\chi'(G)$ of a graph is the least number of colours needed to colour the edges of G in such a way that no two adjacent edges are assigned the same colour.

D1.2.69 A graph G is said to be of class 1 if $\chi'(G) = \rho$, and of class 2 if $\chi'(G) = \rho + 1$, where ρ is the maximum degree of G.

<u>D1.2.70</u> G is said to be critical (edge colouring) if it is connected and of class 2, and if the removal of any edge of G lowers its chromatic index.

<u>D1.2.71</u> We denote by Cr(G) the subgraph of G spanned by the set of critical lines in G. If Cr(G) = G, then G is line-critical.

<u>D1.2.72</u> The symbol // marks the end of a proof.

CHAPTER 2

SOME PROPERTIES OF CRITICAL GRAPHS

CHAPTER 2

Some Properties of Critical Graphs

2.1 Introduction

Berge's [11] strong perfect graph conjecture (SPGC) asserts that a critical graph is either an odd chordless cycle of length greater than or equal to five or the complement of such a cycle. Hence a characterization of critical graphs plays an important part to settle Berge's conjecture. With this in view, we obtain some new properties of critical graphs which, we hope, will be a step towards the solution of the conjecture.

A few results obtained for critical graphs include a relationship between the parameters α, Θ and ω , χ ; lower and upper bounds of $\omega+\alpha$ and $|H_1| \cdot |H_2|$ and expressions for $\Theta(H_2(v))$ and $\chi(H_1(v))$ in terms of Θ and χ respectively. We prove that the minimum number of vertices of a critical graph of even order, if it exists, is twentysix. Among all graphs up to 16 vertices the only critical graphs are C_{2n+1} and $\overline{C_{2n+1}}$, $2 \leq n \leq 7$. We establish that the SPGC is true for $\overline{K_{1,3}}$ -free graphs.

We start with some results already available. Some of these results throw light on degree constraints of the vertices, bounds on the parameters ω and α and the nature of subgraphs like $H_1, H_2, G-Q$ and $G-S$ of critical graphs. Sachs [106] obtained the following properties of critical graphs :

a) If G is critical, then

$$\alpha \geq 2 \tag{2.1.1}$$

b) For any complete subgraph Q in G,

$$\alpha(G-Q) = \alpha \tag{2.1.2}$$

In particular,

$$\alpha(G-v) = \alpha \tag{2.1.3}$$

c) For any $v \in V$, there are at least α distinct maximum independent sets containing v (2.1.4)

d) For any $v \in V$,

$$\alpha(H_2(v)) = \alpha - 1 \tag{2.1.5}$$

e) Each member of a θ-cover for $H_2(v)$ contains at least two points (2.1.6)

f) For any $v \in V$,

$$d(v) \leq p-2\alpha+1 \tag{2.1.7}$$

g) For any $v \in V$, $H_2(v)$ is a connected graph (2.1.8)

Parthasarathy and Ravindra [92] showed a few more properties of critical graphs. They are as follows :

i) If G is critical, then

$$\omega \geq 2 \tag{2.1.9}$$

ii) For any independent set S of G,

$$\omega(G-S) = \omega \tag{2.1.10}$$

In particular,

$$\omega(G-v) = \omega \quad \text{for any } v \in V \tag{2.1.11}$$

iii) For any $v \in V$, there are at least ω maximum cliques containing v. (2.1.12)

iv) For any $v \in V$,

$$\omega(H_1(v)) = \omega - 1 \tag{2.1.13}$$

v) For any $v \in V$,

$$d(v) \geq 2\omega - 2 \tag{2.1.14}$$

vi) $\overline{G}$ is critical (2.1.15)

vii) $\omega\alpha = p-1$ (2.1.16)

viii) If G is critical and $\omega = 2$, then

$$G = C_{2n+1}, \; n \geq 2 \tag{2.1.17}$$

ix) If G is critical and $\alpha = 2$, then

$$G = \overline{C_{2n+1}}, \; n \geq 2 \tag{2.1.18}$$

x) If G is a $K_{1,3}$ -free critical graph, then G is C_{2n+1} or $\overline{C_{2n+1}}$, $n \geq 2$ (2.1.19)

xi) If G is critical and $K_{1,3}$ -free, then $\overline{G}$ is also critical and $K_{1,3}$ -free. (2.1.20)

We will use the following results of Lovasz ([71], [72]).

1) A graph G is α-perfect if and only if it is χ-perfect. (2.1.21)

2) A graph G is perfect if and only if $\omega(H)\cdot\alpha(H) \geq |H|$, for every induced subgraph H of G. (2.1.22)

2.2 Some results on critical graphs

First of all, we show that there exists no critical graph with the property that $\alpha < \theta$ and $\omega = \chi$ or $\alpha = \theta$ and $\omega < \chi$.

THEOREM 2.2.1 For all critical graphs, $\alpha < \theta$ and $\omega < \chi$.

Proof : For any graph G we have the following inequalities [13].

$$\alpha \leq \theta \tag{2.2.1}$$

$$\omega \leq \chi \tag{2.2.2}$$

We prove that there is no critical graph with the following properties :

$$\alpha = \theta \quad \text{and} \quad \omega = \chi \tag{2.2.3}$$

$$\alpha < \theta \quad \text{and} \quad \omega = \chi \tag{2.2.4}$$

$$\alpha = \theta \quad \text{and} \quad \omega < \chi \tag{2.2.5}$$

Let G be a critical graph with the property (2.2.3). Since G is critical, G-v is perfect for every $v \varepsilon V$. If $\alpha = \theta$ and $\omega = \chi$, then G is a perfect graph, thus contradicting the hypothesis.

If possible, let G be a critical graph satisfying (2.2.4). Again G-v is perfect for all $v \varepsilon V$. Hence

$$\omega(H) = \chi(H)$$

for every induced subgraph H of G-v. In addition, if

$$\omega(G) = \chi(G),$$

then G is a χ-perfect graph and hence a perfect graph by (2.1.21), contradicting the criticality of G.

By an exactly similar argument we can show that there is no critical graph satisfying (2.2.5). //

The next result exhibits a relation between the parameters α and θ and ω and χ for a critical graph.

<u>THEOREM 2.2.2</u> For all critical graphs

$$\theta = \alpha + 1$$

and $$\chi = \omega + 1.$$

<u>Proof</u> : Let G be a critical graph and let v be any vertex of G. G-v is a perfect graph. Hence,

$$\alpha(H) = \theta(H)$$

for every induced subgraph H of G-v. In particular,

$$\alpha(G-v) = \Theta(G-v) \tag{2.2.6}$$

$$\alpha(G-v) = \alpha,$$

by (2.1.3). That means α cliques are enough to cover all the vertices of $G-v$. (2.2.7)

Now consider the covering number of G,

$$\Theta \geq \alpha ,$$

by (2.2.1). With the help of (2.2.7) we can see that Θ cannot exceed $\alpha+1$. Because in the worst case we may require one more clique in order to cover the vertex v than the number of cliques needed for covering all the vertices of $G-v$. Hence the covering number of G may be any one of the following :

$$\Theta(G) = \alpha \tag{2.2.8}$$

$$\Theta(G) = \alpha + 1 \tag{2.2.9}$$

The possibility (2.2.8) will lead to a contradiction. Because in that case

$$\alpha(H) = \Theta(H)$$

for every induced subgraph H of $G-v$, and in addition

$$\alpha(G) = \Theta(G) .$$

Combining these facts and observing that each proper induced subgraph of G is perfect we get

$$\alpha(H) = \Theta(H)$$

for every induced subgraph H of G. So the graph G is α-perfect and hence perfect (2.1.21), contradicting the hypothesis. Therefore, the only possibility is

$$\theta = \alpha + 1.$$

An argument similar to the one given above may be used to establish that for all critical graphs

$$\chi = \omega + 1. \ //$$

Theorem 2.2.2 gives only a necessary condition for a graph to be critical. It is not sufficient. For example, consider the square of a cycle of length eight given in Fig. 2.2.1. It has

$$\alpha = 2,$$

$$\theta = 3,$$

$$\omega = 3,$$

and

$$\chi = 4.$$

$C_8^{\,2}$ is not critical because it has induced cycles of length five like $v_1\ v_2\ v_4\ v_5\ v_7\ v_1$.

<u>COROLLARY 2.2.1</u> For all critical graphs, $\chi(H_1(v)) = \chi - 2$.

<u>Proof</u> : We have from (2.1.13) that

$$\omega(H_1(v)) = \omega - 1.$$

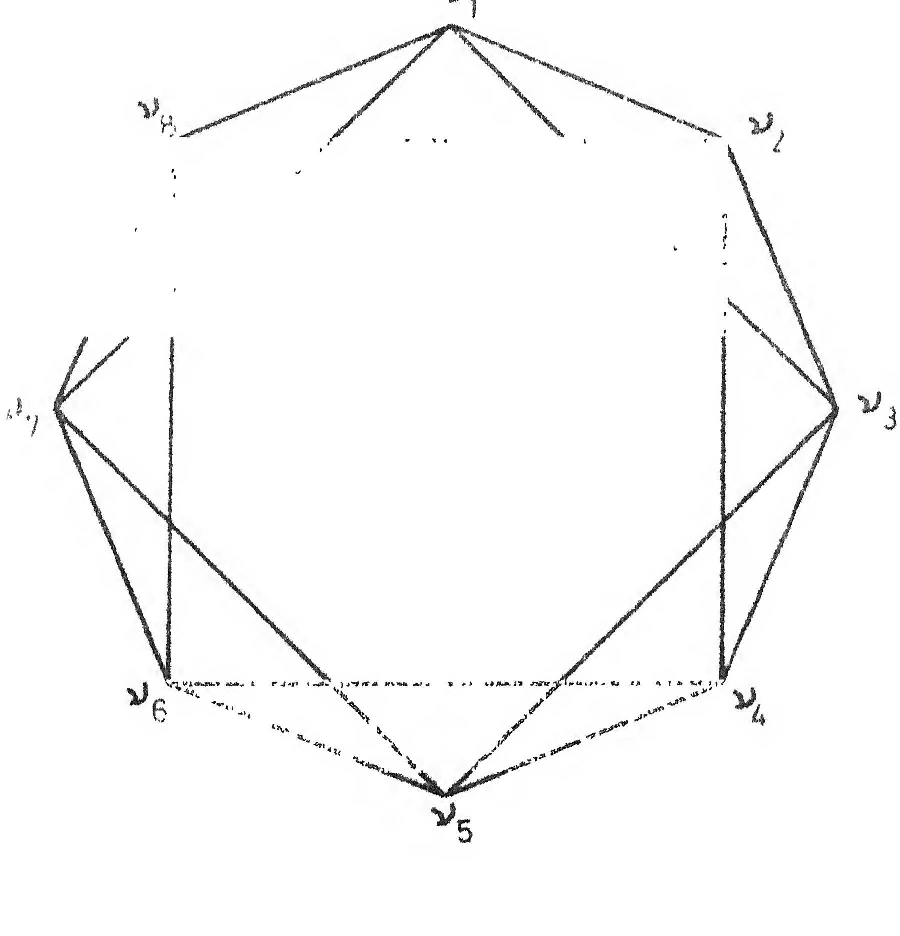


Fig. 2.2.1

Since $H_1(v)$ is a perfect graph

$$\omega(H_1(v)) = \chi\ (H_1(v))\ . \qquad (2.2.10)$$

Therefore,

$$\chi(H_1(v)) = \omega - 1. \qquad (2.2.11)$$

Now substituting for ω in terms of χ from Theorem 2.2.2 we get the required result. //

The next result expresses the covering number of the induced subgraph H_2 of a critical graph G in terms of the covering number of G.

<u>THEOREM 2.2.3</u> For all critical graphs, $\theta(H_2(v)) = \theta - 2$.

<u>Proof</u> : Let G be a critical graph and let v be any vertex of G. From (2.1.3) we see that no critical graph has an isolated vertex. Hence

$$N(v) \neq \varphi \qquad (2.2.12)$$

Therefore G - N(v) is a perfect graph. So

$$\alpha(G-N(v)) = \theta(G-N(v))\ . \qquad (2.2.13)$$

There are α distinct maximum independent sets containing v by (2.1.4). None of these has a vertex common with N(v). So that

$$\alpha(G-N(v)) \geq \alpha\ . \qquad (2.2.14)$$

Since G-N(v) is an induced subgraph of G,

$$\alpha(G-N(v)) \leq \alpha \ , \tag{2.2.15}$$

because $\{u_1, u_2, \ldots, u_t\}$,

$$t > \alpha \tag{2.2.16}$$

forms an independent set in $G-N(v)$ would imply that the same set is independent in G, contradicting the fact that α is the independence number of G. Combining (2.2.14) and (2.2.15) we get

$$\alpha(G-N(v)) = \alpha \ . \tag{2.2.17}$$

Hence,

$$\theta(G-N(v)) = \alpha$$

by (2.2.13). Now substituting for α in terms of θ from Theorem 2.2.2 we get

$$\theta(G-N(v)) = \theta - 1 \ . \tag{2.2.18}$$

That means that

$$\theta(\langle \overline{N}(v) \cup \{v\} \rangle) = \theta - 1. \tag{2.2.19}$$

But v is not adjacent to any vertex of $\overline{N}(v)$. Therefore no clique of $\langle \overline{N}(v) \rangle$ can cover v. So

$$\theta(H_2(v)) = \theta - 2 \ .//$$

<u>COROLLARY 2.2.2</u> (Sachs [106]) For critical graphs

$$\alpha(H_2(v)) = \alpha - 1 \ .$$

Proof : $H_2(v)$ is a perfect graph. Hence

$$\alpha(H_2(v)) = \theta(H_2(v)) \ . \tag{2.2.20}$$

$$\theta(H_2(v)) = \theta - 2$$

by Theorem 2.2.3. Now substituting for θ in terms of α from Theorem 2.2.2 we get the result .//

LEMMA 2.2.1 For all critical graphs

$$2 \leq \omega \leq \frac{p-1}{2}$$

$$2 \leq \alpha \leq \frac{p-1}{2}$$

Proof : The lower bound of ω and α are given by (2.1.9) and (2.1.1) respectively. All critical graphs have $\omega\alpha = p-1$ by (2.1.16). The upper bound of ω is obtained by substituting the lower bound of α in this equation. Therefore

$$\omega \leq \frac{p-1}{2} \ .$$

Similarly, $\alpha \leq \frac{p-1}{2}$. //

The next result gives the upper and lower bounds for the sum of ω and α.

LEMMA 2.2.2 For critical graphs

$$2(p-1)^{\frac{1}{2}} \leq \omega + \alpha \leq \frac{p+3}{2} \ .$$

Proof : In order to get the upper bound of $\omega + \alpha$, we combine (2.1.14) and (2.1.7). That is,

$$2\omega \;\; -2 \;\leq\; d(v) \;\leq\; p-2\alpha+1 \;. \qquad (2.2.21)$$

Hence

$$2\omega \;\; -2 \;\leq\; p-2\alpha+1 \;. \qquad (2.2.22)$$

Therefore,

$$\omega+\alpha \;\leq\; \frac{p+3}{2} \;. \qquad (2.2.23)$$

The lower **bound** is obtained by using the following principle.

Suppose that x and y are two positive integers such that their product is a constant, say k.

That is,

$$xy = k \qquad (2.2.24)$$

Then

$$\tfrac{1}{4}[(x+y)^2 - (x-y)^2] = k \qquad (2.2.25)$$

This implies $(x+y)^2 = 4k + (x-y)^2$. (2.2.26)

The minimum value of $(x+y)^2$ is obtained when $x = y = k^{\frac{1}{2}}$, from the right hand side of equation (2.2.26). Hence the minimum value of x+y is $2k^{\frac{1}{2}}$.

From (2.1.16) we have

$$\omega\alpha = p - 1$$

Therefore, applying the above principle,

$$\omega + \alpha \;\geq\; 2(p-1)^{\frac{1}{2}} \;. \;//$$

Berge's SPGC implies that a critical graph is either a hole or an antihole. That means the order of a critical graph

is odd. The following theorem gives the minimum number of vertices of a critical graph of even order, if at all such a graph exists.

<u>THEOREM 2.2.4</u> The minimum number of vertices of a critical graph of even order, if it exists, is twentysix.

<u>Proof</u> : From (2.1.16), we have for a critical graph,

$$\omega\alpha = p-1 \ .$$

$$\alpha \geq 2$$

by (2.1.1) and

$$\omega \geq 2 \ ,$$

by (2.1.9). Therefore, if p-1 is a prime number it is not possible to find an ω and α satisfying (2.1.1) and (2.1.9). This implies that there cannot exist a critical graph whose order is q+1, where q is a prime number. As a result, there is no critical graph with vertices 2,4,6,8,12,14,18,20 or 24.

<u>Case (i)</u> We shall prove that there exists no critical graph with ten vertices. Assume that G is a critical graph with 10 vertices. Then

$$\omega\alpha = 9 \ . \tag{2.2.27}$$

by (2.1.16). So,

$$\omega(G) = 3 \ . \tag{2.2.28}$$

G is not a hole or an antihole. Since G is a critical graph

it does not contain a hole or an antihole as an induced subgraph. Hence by ([125], Theorem 9) namely; if G has no holes or antiholes and $\omega(G) = 3$ then G is perfect, we get G to be a perfect graph, contradicting the fact that G is a critical graph.

<u>Case (ii)</u> Now we show that a critical graph with 16 vertices does not exist. Suppose that there is a critical graph G_1 with 16 vertices. Then

$$\omega(G_1) \cdot \alpha(G_1) = 15 . \tag{2.2.29}$$

Now there are two possibilities :

$$\omega(G_1) = 3 \quad \text{and} \quad \alpha(G_1) = 5 . \tag{2.2.30}$$

$$\omega(G_1) = 5 \quad \text{and} \quad \alpha(G_1) = 3 . \tag{2.2.31}$$

If G_1 satisfies (2.2.30) then G_1 is a perfect graph since it is not a hole or an antihole and $\omega(G_1) = 3$.

In the second case the graph $\overline{G_1}$ is perfect because

$$\omega(\overline{G_1}) = \alpha(G_1) = 3.$$

Hence G_1 is also a perfect graph [71].

<u>Case (iii)</u> To establish that there is no critical graph with 22 vertices, we can use a similar argument and prove it . //

<u>COROLLARY 2.2.3</u> For a critical graph of even order, if it exists

$$4 \leq \omega \leq \frac{p-1}{4} \text{ and}$$

$$4 \leq \alpha \leq \frac{p-1}{4}$$

Proof : For a critical graph G of even order p, p-1 is odd. Since

$$\omega\alpha = p-1$$

for all critical graphs, ω and α are odd. Hence for G,

$$\omega \geq 3 . \tag{2.2.32}$$

and

$$\alpha \geq 3 . \tag{2.2.33}$$

A graph G having no holes or antiholes and with $\omega = 3$ is perfect ([125], Theorem 9). Therefore,

$$\omega(G) \geq 4 . \tag{2.2.34}$$

There is no critical graph G_1 of even order with $\alpha(G_1) = 3$. Because in that case $\overline{G_1}$ would be a perfect graph since

$$\omega(\overline{G_1}) = \alpha(G_1) = 3.$$

So

$$\alpha(G) \geq 4 . \tag{2.2.35}$$

Now the upper bound of ω and α follows by using

$$\omega\alpha = p-1 . //$$

COROLLARY 2.2.4 Among all graphs up to 16 vertices the only critical graphs are C_{2n+1} and $\overline{C_{2n+1}}$, $2 \leq n \leq 7$.

Proof : We have seen in Theorem 2.2.4 that there is no critical graph with 2,4,6,8,10,12,14 or 16 vertices.

The graph with a single vertex is trivially perfect. From (2.1.9) and (2.1.14) we see that the minimum degree of a vertex of a critical graph is 2. Hence three of the 4 graphs on 3 vertices [56] do not qualify to be a critical graph. The remaining graph is the complete graph on 3 vertices which is trivially perfect.

From Lemma 2.2.1 we have for critical graphs

$$2 \leq \omega \leq \frac{p-1}{2} . \tag{2.2.36}$$

$$2 \leq \alpha \leq \frac{p-1}{2} . \tag{2.2.37}$$

The critical graph with $\omega = 2$ is C_{2n+1}, $n \geq 2$ by (2.1.17) and the critical graph with $\alpha = 2$ is $\overline{C_{2n+1}}$, $n \geq 2$ by (2.1.18). For a critical graph on 5 vertices

$$\omega\alpha = 4, \tag{2.2.38}$$

by (2.1.16). Hence

$$\omega = 2 . \tag{2.2.39}$$

and

$$\alpha = 2 . \tag{2.2.40}$$

By (2.1.17) $\omega = 2$ corresponds to C_5 and by (2.1.18) $\alpha = 2$ corresponds to $\overline{C_5}$. Since

$$\overline{C_5} = C_5 ,$$

C_5 is the only critical graph on 5 vertices.

For a critical graph on 7 vertices ω can take only the values 2 and 3. The value $\omega = 2$ corresponds to the critical graph C_7 by (2.1.17) and the value $\omega = 3$ which is the same as $\alpha = 2$ corresponds to $\overline{C_7}$ by (2.1.18).

A critical graph on 9 vertices will have the values 2,3 or 4 for ω. The value of α corresponding to $\omega = 3$ is not an integer. And $\omega = 2$ and $\omega = 4$ correspond respectively to C_9 and $\overline{C_9}$.

In the case of a critical graph on 11 vertices ω can take values 2,3,4 and 5. The values of α corresponding to $\omega = 3$ and $\omega = 4$ are not integers. The remaining values of ω correspond to C_{11} and $\overline{C_{11}}$.

If we consider a critical graph on 13 vertices, ω can have values 2,3,4,5 or 6. The value of α corresponding to $\omega = 5$ is not an integer. If $\omega = 3$, the graph becomes perfect ([125], Theorem 9). The value $\omega = 4$ gives $\alpha = 3$ and in that case the complement has $\omega = 3$. As a result, the original graph as well as its complement become perfect. The remaining values of ω correspond to C_{13} and $\overline{C_{13}}$.

The only critical graphs on 15 vertices are C_{15} and $\overline{C_{15}}$ as the values $\omega = 3,4,5,6$ all give nonintegral values of α. //

Let p_1 and p_2 represent respectively the order of $H_1(v)$ and $H_2(v)$. We shall now find the lower and upper bounds of the numbers p_1, p_2 and $p_1 \cdot p_2$.

LEMMA 2.2.3 For critical graphs

$$2 \leq p_1 \leq p-3$$

$$2 \leq p_2 \leq p-3$$

Proof : Combining (2.1.9) and (2.1.14) namely

$$d(v) \geq 2\omega - 2$$

$$\omega \geq 2$$

we have

$$p_1 \geq 2 . \qquad (2.2.41)$$

Applying this to the complement since the complement of a critical graph is critical (2.1.15) we get

$$d(\overline{G},v) \geq 2\,\omega(\overline{G}) - 2$$

$$\omega(\overline{G}) \geq 2 .$$

Hence

$$d(\overline{G},v) \geq 2 . \qquad (2.2.42)$$

But

$$d(\overline{G},v) = |H_1(\overline{G},v)|$$

and

$$|H_1(\overline{G},v)| = |H_2(G,v)| . \qquad (2.2.43)$$

Therefore

$$p_2 \geq 2 . \qquad (2.2.44)$$

Now the upper bound of p_1 and p_2 follows from the fact that for any graph G,

$$p_1 + p_2 = p - 1 , \qquad (2.2.45)$$

and for critical graphs $p_1 \geq 2$; $p_2 \geq 2$ by (2.2.41) and (2.2.44) . //

<u>LEMMA 2.2.4</u> For critical graphs

$$2(p-3) \leq p_1 \cdot p_2 \leq (\tfrac{p-1}{2})^2 .$$

<u>Proof</u> : Let x and y be two positive numbers such that their sum is a constant, say k.

That is $$x + y = k \qquad (2.2.46)$$

so that $$xy = \tfrac{1}{4}[k^2 - (x-y)^2] \qquad (2.2.47)$$

From (2.2.47) we see that xy is maximum when $(x-y)^2$ is minimum. The minimum value of $(x-y)^2$ is attained when $x = y = \frac{k}{2}$. Therefore,

$$xy \leq (\tfrac{k}{2})^2 . \qquad (2.2.48)$$

The least value of xy corresponds to the largest value of $(x-y)^2$. Hence the minimum value of xy is obtained when the difference between x and y is maximum.

Applying these observations to the numbers p_1 and p_2 and using $p_1 + p_2 = p - 1$, we see that

$$p_1 \cdot p_2 \leq (\tfrac{p-1}{2})^2 . \qquad (2.2.49)$$

The maximum difference between the numbers p_1 and p_2 is achieved when one of them is p-3 and the other is 2, by Lemma 2.2.3.

Hence $\qquad p_1 \cdot p_2 \geq 2(p-3)$. //

2.3 Critical $\overline{K_{1,3}}$ -free graphs

In this section we shall find a few properties of critical and $\overline{K_{1,3}}$ -free graphs and establish that Berge's SPGC is valid for $\overline{K_{1,3}}$ -free graphs.

THEOREM 2.3.1 If a graph G is critical and $\overline{K_{1,3}}$ -free, then $d(v) = p+1-2\alpha$.

Proof : Let G be critical and a $\overline{K_{1,3}}$ -free graph. By definition no vertex of $H_2(v)$ is adjacent to v. If $u_1\, u_2\, u_3\, u_1$ is a triangle in $H_2(v)$, then $\langle\{u_1, u_2, u_3, v\}\rangle$ is $\overline{K_{1,3}}$. As a result $H_2(v)$ is a triangle-free graph. Therefore

$$\omega(H_2(v)) \leq 2 . \tag{2.3.1}$$

But $H_2(v)$ is a connected graph by (2.1.8), having at least two vertices by Lemma 2.2.3. Hence

$$\omega(H_2(v)) \geq 2 . \tag{2.3.2}$$

So

$$\omega(H_2(v)) = 2 . \tag{2.3.3}$$

We have for a critical graph

$$\alpha(H_2(v)) = \alpha-1 ,$$

by (2.1.5). Now $H_2(v)$ is a perfect graph. Applying (2.1.22)

$$2(\alpha-1) \geq |H_2| . \tag{2.3.4}$$

From (2.1.15) and (2.1.14) we get

$$d(\overline{G},v) \geq 2\omega(\overline{G}) - 2.$$

That is, $|H_1(\overline{G},v)| \geq 2\alpha(G) - 2,$

Since $\omega(\overline{G}) = \alpha(G)$.

Hence $|H_2(G,v)| \geq 2\alpha - 2$, (2.3.5)

using (2.2.43).

Combining (2.3.4) and (2.3.5) we get

$$|H_2(v)| = 2\alpha - 2.$$

Evidently $|H_1(v)| = p-1 - (2\alpha-2),$

using (2.2.45).

That means $d(v) = p+1 - 2\alpha$. //

<u>COROLLARY 2.3.1</u> For a critical and $K_{1,3}$ -free graph G, $\alpha(H_1(G,v)) = 2$.

<u>Proof</u> : First of all we shall prove that if a graph G is $K_{1,3}$ -free, then $\overline{G}$ is $\overline{K_{1,3}}$ -free. If possible let $\overline{G}$ possess a $\overline{K_{1,3}}$ say, on u_1,u_2,u_3 and v. Evidently $<\{u_1,u_2,u_3,v\}>$ is a $K_{1,3}$ in G. This goes against the hypothesis. Therefore, if a graph G is critical and $K_{1,3}$ -free, then $\overline{G}$ is critical and $\overline{K_{1,3}}$ -free and conversely. The graphs $H_2(\overline{G},v)$ and $H_1(G,v)$

are complementary. Hence

$$\omega(H_2(\overline{G},v)) = \alpha(H_1(G,v)) \tag{2.3.6}$$

Since $\overline{G}$ is critical and a $\overline{K_{1,3}}$ -free graph

$$\omega(H_2(\overline{G},v)) = 2.$$

by (2.3.3). Therefore

$$\alpha(H_1(G,v)) = 2,$$

by (2.3.6) . //

<u>COROLLARY 2.3.2</u> [92] If a graph G is critical and $K_{1,3}$ -free then $d(G,v) = 2\omega(G) - 2$.

<u>Proof</u> : The graph $\overline{G}$ is critical and $\overline{K_{1,3}}$ -free. Hence by Theorem 2.3.1

$$d(\overline{G},v) = p+1 - 2\alpha(\overline{G}) \tag{2.3.7}$$

Therefore $$|H_1(\overline{G},v))| = p+1 - 2\omega(G) \tag{2.3.8}$$

So $$|H_2(G,v))| = p+1 - 2\omega(G) \tag{2.3.9}$$

Now using (2.2.45)

$$|H_1(G,v)| = p-1-(p+1-2\omega(G)).$$

This implies

$$d(G,v) = 2\omega(G) - 2 \ . //$$

<u>LEMMA 2.3.1</u> Every critical and $\overline{K_{1,3}}$ -free graph is either a hole or an antihole.

Proof : If G is critical and a $K_{1,3}$ -free graph, then G is C_{2n+1} or $\overline{C_{2n+1}}$, $n \geq 2$ by (2.1.19).

Consider any critical and $\overline{K_{1,3}}$ -free graph G_1.

$$\overline{G_1} \text{ is critical and } K_{1,3} \text{ -free.} \tag{2.3.10}$$

Hence $\overline{G_1}$ is C_{2n+1} or $\overline{C_{2n+1}}$, $n \geq 2$. (2.3.11)

This implies G_1 is either C_{2n+1} or $\overline{C_{2n+1}}$, $n \geq 2$. //

THEOREM 2.3.2 Berge's SPGC is true for $\overline{K_{1,3}}$ -free graphs.

Proof : If possible, let G be an imperfect graph which does not contain $\overline{K_{1,3}}$ or C_{2n+1} or $\overline{C_{2n+1}}$, $n \geq 2$, as an induced subgraph. Let H be an imperfect induced subgraph of G with minimum number of vertices. Then

$$H \text{ is critical and } \overline{K_{1,3}} \text{ -free.} \tag{2.3.12}$$

Hence by Lemma 2.3.1

$$H \text{ is } C_{2n+1} \text{ or } \overline{C_{2n+1}},\ n \geq 2 , \tag{2.3.13}$$

contradicting the hypothesis. //

CHAPTER 3

B-GRAPHS AND B*-GRAPHS

CHAPTER 3

B-graphs and B^*-graphs

3.1 Introduction

This chapter deals with some properties of B-graphs, B^*-graphs and B^{**}-graphs. The concept of B-graphs was introduced to study the critical graphs. From (2.1.4) we see that one fundamental property of critical graphs is that each vertex of the graph is in a maximum independent set of vertices. So researchers were trying to characterize graphs with this property keeping in view that critical graphs belong to this class of graphs. A graph is a B-graph if each vertex is in a maximum independent set of vertices. Analogously we define B^*-graphs to be graphs where each edge is in a maximum independent set of edges of the graph. A graph which is both a B-graph and a B^*-graph is called a B^{**}-graph. We give necessary and sufficient conditions for a path, the join of two B-graphs and an acyclic graph without isolated vertices to be a B-graph. An algorithm is suggested to generate new B-graphs from known ones. Powers of cycles and line graphs of powers of cycles are proved to be B^{**}-graphs. It is shown that we need not search for B^{**}-graphs among acyclic graphs.

3.2 B-graphs

The independence number is additive on the components of a graph [13]. Hence an obvious consequence of the definition of B-graphs is the following Lemma.

LEMMA 3.2.1 Union of B-graphs is a B-graph.

LEMMA 3.2.2 The independence number of a path P_n is $[\frac{n+1}{2}]$.

Proof : The number n of vertices in a path is either even or odd.

Case (i) Let n be an odd integer, say 2t-1. Let $v_1, v_2, \ldots, v_{2t-1}$ be the vertices and $e_i = (v_i, v_{i+1})$, $1 \leq i \leq 2t-2$ be the edges of P_{2t-1}. The set

$$S = \{v_1, v_3, \ldots, v_{2t-1}\} \tag{3.2.1}$$

is an independent set of vertices.

Hence

$$\alpha(P_{2t-1}) \geq t . \tag{3.2.2}$$

$$\beta_o(P_{2t-1}) \leq t \tag{3.2.3}$$

since the number of vertices is 2t-1. For bipartite graphs we have,

$$\alpha = \beta_o \tag{3.2.4}$$

by [6]. Now combining (3.2.2), (3.2.3) and (3.2.4) we get that $\alpha(P_{2t-1}) = t$.

Case (ii) Suppose that n is an even integer, say 2t. Let $v_1, v_2, \ldots, v_{2t}$ be the vertices and $e_i = (v_i, v_{i+1})$, $1 \leq i \leq 2t-1$ be the edges of P_{2t}. The matching

$$M = \{e_1, e_3, e_5, \ldots, e_{2t-1}\} \tag{3.2.5}$$

is a perfect matching of P_{2t} consisting of t edges. Hence

$$\beta(P_{2t}) = t . \tag{3.2.6}$$

Therefore

$$\beta_o(P_{2t}) = t \tag{3.2.7}$$

by [43]. P_{2t} is a bipartite graph. Hence from (3.2.7) and (3.2.4), we get that

$$\alpha(P_{2t}) = t . //$$

THEOREM 3.2.1 A path is a B-graph if and only if it is P_{2n}.

Proof : First we show that P_{2n} is a B-graph. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices and $e_i = (v_i, v_{i+1})$, $1 \leq i \leq 2n-1$ be the edges of P_{2n}. Let

$$S_1 = \{v_1, v_3, \ldots, v_{2n-1}\} \tag{3.2.8}$$

and

$$S_2 = \{v_2, v_4, \ldots, v_{2n}\}. \tag{3.2.9}$$

S_1 and S_2 are independent sets containing n elements each.

$$\alpha(P_{2n}) = n \tag{3.2.10}$$

by Lemma 3.2.2.

$$S_1 \cup S_2 = V . \tag{3.2.11}$$

Hence P_{2n} is a B-graph.

It is enough to show that a path of even length is not a B-graph. Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices and $e_i = (v_i, v_{i+1})$, $1 \leq i \leq 2n$ be the edges of P_{2n+1}.

$$\alpha(P_{2n+1}) = n+1 \tag{3.2.12}$$

by Lemma 3.2.2. We shall prove that there exists no vertex of P_{2n+1} with an even suffix which is in an independent set containing n+1 vertices. Consider an independent set containing v_2. The vertices v_1 and v_3 cannot appear in this set. All the remaining vertices of P_{2n+1} lie on P_{2n-2}. Hence all the vertices for an independent set containing v_2 are to be selected from $V(P_{2n-2})$. Therefore the maximum order of an independent set involving v_2 is n since

$$\alpha(P_{2n-2}) = n-1$$

by Lemma 3.2.2. Similarly v_{2n} is not in a MISS. In the case of v_{2t}, $2 \leq t \leq n-1$, the vertices v_{2t-1} and v_{2t+1} cannot be present in an independent set to which v_{2t} belongs. The remaining vertices of P_{2n+1} lie on paths P_{2r} and P_{2s} so that

$$2r + 2s = 2n-2 . \tag{3.2.13}$$

As a result, the maximum order of an independent set for which v_{2t} is a member is n. //

LEMMA 3.2.3 Independence number of a cycle C_n is $[\frac{n}{2}]$.

Proof : We discuss three cases.

Case (i) C_3 is a complete graph. Hence

$$\alpha(C_3) = 1.$$

Case (ii) C_{2t+1}, $t \geq 2$ is a critical graph with

$$\omega(C_{2t+1}) = 2 , \tag{3.2.14}$$

by [50]. For a critical graph having p vertices

$$\omega\alpha = p-1$$

by (2.1.16). Therefore

$$2\alpha(C_{2t+1}) = 2t \tag{3.2.15}$$

Hence $$\alpha(C_{2t+1}) = t .$$

Case (iii) Let $v_1, v_2, \ldots, v_{2t}$ be the vertices and $e_i = (v_i, v_{i+1})$ (suffix modulo 2t) be the edges of C_{2t}. The sets

$$V_1 = \{v_1, v_3, v_5, \ldots, v_{2t-1}\} \tag{3.2.16}$$

and

$$V_2 = \{v_2, v_4, v_6, \ldots, v_{2t}\} \tag{3.2.17}$$

are independent sets of order t. Hence C_{2t} is a bipartite graph and

$$\alpha(C_{2t}) \geq t \tag{3.2.18}$$

The edges of the set

$$M = \{e_1, e_3, e_5, \ldots, e_{2t-1}\} \tag{3.2.19}$$

cover all the vertices of C_{2t}. Therefore

$$\beta_o(C_{2t}) \leq t . \tag{3.2.20}$$

Now combining (3.2.4), (3.2.18) and (3.2.20), we get $\alpha(C_{2t}) = t$. //

<u>COROLLARY 3.2.1</u> C_{2t} is a B-graph.

<u>Proof</u> : From [103] we know that a bipartite graph is a B-graph if and only if

$$|V_1| = |V_2| = \alpha(G)$$

C_{2t} satisfies this condition .//

<u>THEOREM 3.2.2</u> Every cycle is a B-graph.

<u>Proof</u> : Let $v_1, v_2, v_3, \ldots, v_{2t+1}, v_1$ be the cycle C_{2t+1}. Consider the sets V_1, V_2 given in (3.2.16) and (3.2.17) and the set

$$V_3 = \{v_3, v_5, v_7, \ldots, v_{2t+1}\}. \tag{3.2.21}$$

Each of these is an independent set of C_{2t+1} of order t. This implies that each vertex of C_{2t+1} is in a MISS.

The fact that C_{2t} is a B-graph is proved in Corollary 3.2.1 . //

<u>THEOREM 3.2.3</u> An acyclic graph without isolated vertices is a B-graph if and only if it has a perfect matching.

Proof : Suppose that an acyclic graph T has a perfect matching. T is a bipartite graph by Königs Theorem [23]. Hence by (3.2.4)

$$\alpha_o(T) = \beta(T).$$

The order of a graph with a perfect matching is even [6]. This implies that

$$\beta(T) = \frac{p}{2} . \tag{3.2.22}$$

For any graph the sum of the point independence number and the point covering number equals the order of the graph [43]. Therefore

$$\alpha + \alpha_o = p . \tag{3.2.23}$$

Now substituting the value of $\alpha_o(T)$ in (3.2.23) gives

$$\alpha(T) = \frac{p}{2} . \tag{3.2.24}$$

This means that each of the sets V_1, V_2 which partition $V(T)$ is of order $p/2$. Hence T is a B-graph.

Consider an acyclic B-graph G without isolated vertices. G is a bipartite graph. Hence by [103]

$$\alpha(G) = p/2 \tag{3.2.25}$$

So

$$\alpha_o(G) = p/2 \tag{3.2.26}$$

Since

$$\alpha_o = \beta$$

for bipartite graphs, we have

$$\beta(G) = p/2 \tag{3.2.27}$$

Hence there exists a matching having p/2 edges and this matching is a perfect matching. //

THEOREM 3.2.4 C_p^r, $1 \leq r \leq [p/2]$ is a B-graph.

Proof : We consider C_{2n}^r and C_{2n+1}^r separately.

Case (i) Let $v_1, v_2, v_3, \ldots, v_{2n+1}, v_1$ be the cycle C_{2n+1}. Any vertex v_i in C_{2n+1}^r, $(1 \leq r \leq n)$ is adjacent to

$$v_{i+1}, v_{i+2}, \ldots, v_{i+r} \quad \text{and}$$

$$v_{i+2n}, v_{i+2n-1}, \ldots, v_{i+2n-(r-1)} \text{ (suffix modulo } 2n+1)$$

Consider $S' = \{v_i, v_{i+r+1}, v_{i+2r+2}, \ldots, v_{i+kr+k}\}$ (3.2.28)

where k is the largest integer such that

$$k < \frac{2n-(r-1)}{r+1} \tag{3.2.29}$$

The suffixes of the vertices of S' are less than i+2n because of (3.2.29). Hence all the vertices of S' are distinct. S' is a MISS since each vertex of C_{2n+1}^r other than that of S' is adjacent to at least one vertex of S'. So every vertex of C_{2n+1}^r is in a MISS consisting of k+1 vertices. Therefore C_{2n+1}^r is a B-graph.

Case (ii) Suppose that $v_1, v_2, v_3, \ldots, v_{2n}, v_1$ is the cycle C_{2n}. Every vertex v_i in C_{2n}^r $(1 \leq r \leq n)$ is adjacent to

$$v_{i+1}, v_{i+2}, \ldots, v_{i+r} \quad \text{and}$$

$$v_{i+2n-1}, v_{i+2n-2}, \ldots, v_{i+2n-r} \text{ (suffix modulo } 2n).$$

In this case also we can prove that S' is a MISS by using an exactly similar argument, provided that k is the largest integer such that

$$k < \frac{2n-r}{r+1} \tag{3.2.30}$$

The restriction on k is to ensure that the vertex v_{i+kr+k} is not adjacent to v_i. Hence C_{2n}^r is also a B-graph . //

<u>THEOREM 3.2.5</u> If G_1 and G_2 are B-graphs, then G_1+G_2 is a B-graph if and only if $\alpha(G_1) = \alpha(G_2)$.

<u>Proof</u> : First we shall prove that if G_1 and G_2 are B-graphs with $\alpha(G_1) = \alpha(G_2)$, then G_1+G_2 is a B-graph. Let

$$\alpha(G_1) = \alpha(G_2) . \tag{3.2.31}$$

Suppose that S_i $(1 \leq i \leq r)$, S_j' $(1 \leq j \leq t)$ be the MISSes of G_1 and G_2 respectively. Then

$$\bigcup_{i=1}^{r} S_i = V(G_1) \tag{3.2.32}$$

$$\bigcup_{j=1}^{t} S_j' = V(G_2) \tag{3.2.33}$$

In G_1+G_2, each vertex of G_1 is adjacent to every vertex of G_2. Hence every MISS of G_1+G_2 is either a MISS of G_1 or that of G_2.

$$V(G_1+G_2) = V(G_1) \bigcup V(G_2) . \tag{3.2.34}$$

Hence G_1+G_2 is a B-graph.

Now we shall prove that if G_1 and G_2 are B-graphs and G_1+G_2 is a B-graph, then

$$\alpha(G_1) = \alpha(G_2).$$

Suppose that this is not true. Then without loss of generality we can assume that

$$\alpha(G_1) > \alpha(G_2) \qquad (3.2.35)$$

It follows that no S_j' $(1 \leq j \leq t)$ is a MISS of G_1+G_2. Since in G_1+G_2, each vertex of G_2 is adjacent to all the vertices of G_1, no vertex of G_2 is in a MISS. This contradicts the fact that G_1+G_2 is a B-graph. Hence

$$\alpha(G_1) = \alpha(G_2). \text{ //}$$

We shall use some of these results in Chapter 4 to prove that certain graphs are B-point critical.

<u>Remark 3.2.1</u> Theorem 3.2.5 gives us a method for generating new B-graphs from known B-graphs. For example

$$P_4,\ P_4+P_4,\ P_4+P_4+P_4\ ,\ \ldots \qquad (3.2.36)$$

$$C_n^r,\ C_n^r + C_n^r,\ C_n^r + C_n^r + C_n^r\ ,\ \ldots \qquad (3.2.37)$$

$$C_{2n} + P_{2n},\ C_{2n} + P_{2n} + P_{2n},\ \ldots \qquad (3.2.38)$$

are B-graphs.

3.3 B^*-graphs

A graph G is a B^*-graph if each edge of G is in a maximum independent set of edges of G. Some properties of B^*-graphs are similar to those discussed for B-graphs.

LEMMA 3.3.1 The union of B^*-graphs is a B^*-graph.

The proof follows from the fact that the line independence number of a graph is additive on the components of a graph.

LEMMA 3.3.2 The line independence number of P_n is $[n/2]$.

Proof : The number of vertices in a path is either odd or even.

Case (i) n is an odd integer, say 2t+1.

Since a path is a bipartite graph, we have by (3.2.4)

$$\alpha = \beta_o$$

For any graph the sum of the line independence number and the line covering number of a graph is equal to the order of the graph [43]. That is,

$$\beta + \beta_o = p \; . \qquad (3.3.1)$$

Substituting for α and β_o from (3.2.23) and (3.3.1) in (3.2.4) we get

$$\beta = \alpha_o \; . \qquad (3.3.2)$$

From Lemma 3.2.2 we have

$$\alpha(P_{2t+1}) = t+1 \; . \qquad (3.3.3)$$

Hence

$$\alpha_o(P_{2t+1}) = (2t+1) - (t+1)$$

by (3.2.23).

That is,

$$\alpha_o(P_{2t+1}) = t \tag{3.3.4}$$

Therefore

$$\beta(P_{2t+1}) = t.$$

Case (ii) n is an even integer, say 2t.

From Lemma 3.2.2,

$$\alpha(P_{2t}) = t$$

So

$$\alpha_o(P_{2t}) = t , \tag{3.3.5}$$

by (3.2.23). Hence

$$\beta(P_{2t}) = t$$

substituting in (3.3.2). //

The following Lemma states that an acyclic graph can at most have only one perfect matching. These results are used to prove a necessary and sufficient condition for a path to be a B*-graph.

LEMMA 3.3.3 If an acyclic graph has a perfect matching, then the matching is unique.

Proof : We observe that if a graph G has a perfect matching, it is of even order [6]. Suppose that an acyclic graph G has

two different perfect matchings M_1 and M_2. Let

$$M_1 = \{(u_1,v_1),\ (u_2,v_2),\ \ldots,\ (u_{p/2},\ v_{p/2})\} \qquad (3.3.6)$$

If M_2 is different from M_1, then at least one edge of M_2 is of the form $(u_i,\ v_j)$, $i \neq j$. Hence the edges (v_i,u_i), (u_i,v_j) and (v_j,u_j) form a P_4 in $M_1 \cup M_2$. That means

$$P_4 \text{ is a subgraph of } M_1 \bigcup M_2 . \qquad (3.3.7)$$

But we know that the components of the subgraph produced by the union of two perfect matchings are K_2's and even length cycles. Since the graph G is acyclic,

$$\text{All the components of } M_1 \bigcup M_2 \text{ are } K_2\text{'s.} \qquad (3.3.8)$$

Now (3.3.7) contradicts (3.3.8). Hence G cannot have two different perfect matchings. //

<u>THEOREM 3.3.1</u> A path is a B^*-graph if and only if it is P_{2n+1}.

<u>Proof</u> : Let $v_1,v_2,\ \ldots,\ v_{2n+1}$ be the vertices and

$$e_i = (v_i,\ v_{i+1}),\ 1 \leq i \leq 2n \qquad (3.3.9)$$

be the edges of P_{2n+1}.

The sets

$$M_1' = \{e_1,e_3,e_5,\ \ldots,\ e_{2n-1}\} \qquad (3.3.10)$$

and

$$M_2' = \{e_2,e_4,e_6,\ \ldots,\ e_{2n}\} \qquad (3.3.11)$$

are independent sets having n edges each.

$$M_1' \bigcup M_2' = E . \qquad (3.3.12)$$

Hence P_{2n+1} is a B^*-graph since $\beta(P_{2n+1}) = n$, by Lemma 3.3.2.

Next we prove that P_{2n} is not a B^*-graph. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices and

$$e_i = (v_i, v_{i+1}),\ 1 \leq i \leq 2n-1 \tag{3.3.13}$$

be the edges of P_{2n}. The set

$$M = \{e_1, e_3, e_5, \ldots, e_{2n-1}\} \tag{3.3.14}$$

is an independent set of edges having n elements. M is a perfect matching since it saturates all the 2n vertices. By Lemma 3.3.3 there cannot exist another perfect matching. Hence none of the edges in the set

$$M' = \{e_2, e_4, e_6, \ldots, e_{2n-2}\} \tag{3.3.15}$$

is in a maximum independent set.

Hence P_{2n} is not a B^*-graph. //

We need the following Lemma in order to prove that C_n^r is a B^*-graph.

<u>LEMMA 3.3.4</u> $\beta(C_k^r) = [k/2]$, $(1 \leq r \leq [k/2])$.

<u>Proof</u> : We consider C_{2n+1} and C_{2n} separately.

<u>Case (i)</u> Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices and

$$e_i = (v_i, v_{i+1}),\quad 1 \leq i \leq 2n \tag{3.3.16}$$

be the edges of C_{2n+1}. The set

$$M'' = \{e_1, e_3, e_5, \ldots, e_{2n-1}\} \tag{3.3.17}$$

is an independent set having n edges. C^r_{2n+1} $(1 \leq r \leq n)$ also has M'' as an independent set. Hence

$$\beta(C^r_{2n+1}) \geq n . \tag{3.3.18}$$

The number of vertices in C^r_{2n+1} is 2n+1. Therefore,

$$\beta(C^r_{2n+1}) \leq n. \tag{3.3.19}$$

Combining (3.3.18) and (3.3.19) we get

$$\beta(C^r_{2n+1}) = n$$

<u>Case (ii)</u> We use an argument similar to that given in case (i) to prove that

$$\beta(C^r_{2n}) = n, \ (1 \leq r \leq n) \ . \ //$$

<u>THEOREM 3.3.2</u> If G is an odd cycle with chords, then G is a B^*-graph.

<u>Proof</u> : Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices of G. Let $e_i = (v_i, v_{i+1})$, $1 \leq i \leq 2n+1$ (suffix modulo 2n+1) belong to E(G). Suppose that

$$C_{i,r} = (v_i, v_{i+r}), \ 2 \leq r \leq 2n-1 \tag{3.3.20}$$

(suffix modulo 2n+1) is a chord of G. The sets

$$M_3 = \{e_1, e_3, e_5, \ldots, e_{2n-1}\} \tag{3.3.21}$$

$$M_4 = \{e_2, e_4, e_6, \ldots, e_{2n}\} \tag{3.3.22}$$

$$M_5 = \{e_3, e_5, e_7, \ldots, e_{2n+1}\} \tag{3.3.23}$$

are maximum independent sets since

$$\beta(C_{2n+1}) = n$$

by Lemma 3.3.4. Hence each e_i is in a MISS. We shall prove that $C_{i,r}$ is also in a MISS.

<u>Case (i)</u> Let r be an odd integer, say 2k+1. Consider the following sets.

$$M_6 = \{e_{i+1}, e_{i+3}, e_{i+5}, \ldots, e_{i+2k-1}\} \tag{3.3.24}$$

$$M_7 = \{(v_i, v_{i+2k+1})\} \tag{3.3.25}$$

$$M_8 = \{e_{i+2k+2}, e_{i+2k+4}, e_{i+2k+6}, \ldots, e_{i+2n-2}\} \tag{3.3.26}$$

The set $M_6 \cup M_7 \cup M_8$ is an independent set consisting of n edges since M_6, M_7 and M_8 are disjoint sets of orders k, 1 and n-k-1 respectively.

<u>Case (ii)</u> Suppose that r is an even integer, say 2k. We shall prove that (v_i, v_{i+2k}) is in an independent set of n edges. The sets

$$M_9 = \{e_{i+1}, e_{i+3}, e_{i+5}, \ldots, e_{i+2k-3}\} \tag{3.3.27}$$

$$M_{10} = \{(v_i, v_{i+2k})\} \tag{3.3.28}$$

and

$$M_{11} = \{e_{i+2k+1}, e_{i+2k+3}, e_{i+2k+5}, \ldots, e_{i+2n-1}\} \tag{3.3.29}$$

are disjoint and have orders k-1, 1 and n-k respectively.

Hence $M_9 \cup M_{10} \cup M_{11}$ is an independent set of n edges. Therefore G is a B^*-graph. //

<u>COROLLARY 3.3.1</u> C_{2n+1} is a B^*-graph.

The next two results give conditions so that an even cycle with chords may be a B^*-graph.

<u>THEOREM 3.3.3</u> Let G be an even cycle with chords. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of G. Suppose that $e_i = (v_i, v_{i+1})$, $(1 \leq i \leq 2n)$ and $C_{i,r} = (v_i, v_{i+r})$, $(2 \leq r \leq 2n-2)$ (suffix modulo 2n) are the edges and chords of G respectively. If no $C_{i,r}$ makes the cycle $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+r}, v_i$ which is isomorphic to C_{2t+1}, then G is a B^*-graph.

<u>Proof</u> : The restriction on the chord $C_{i,r}$ supplied in the statement of Theorem 3.3.3 makes r an odd integer; say 2k+1.

The sets

$$M_{12} = \{e_1, e_3, e_5, \ldots, e_{2n-1}\} \tag{3.3.30}$$

and

$$M_{13} = \{e_2, e_4, e_6, \ldots, e_{2n}\} \tag{3.3.31}$$

have n elements each. G has only 2n vertices. Hence M_{12} and M_{13} are maximum independent sets of edges of G. $M_{12} \cup M_{13}$ contains all e_i's. Hence each e_i is in a maximum independent set of edges of G. We shall prove that $C_{i,r}$ also belongs to a maximum independent set. Consider the following sets

$$M_{14} = \{e_{i+1}, e_{i+3}, e_{i+5}, \ldots, e_{i+2k-1}\} \tag{3.3.32}$$

$$M_{15} = \{(v_i, v_{i+2k+1})\} \tag{3.3.33}$$

and

$$M_{16} = \{e_{i+2k+2}, e_{i+2k+4}, e_{i+2k+6}, \ldots, e_{i+2n-2}\} \tag{3.3.34}$$

The edges of M_{14}, M_{15} and M_{16} are independent. The number of elements in these sets are k, 1 and n-k-1 respectively. Therefore, $M_{14} \cup M_{15} \cup M_{16}$ is an independent set containing $C_{i,r}$ with order n. So G is a B^*-graph. //

Figure 3.3.1 gives an illustration of Theorem 3.3.3. In this case

$$M_{14} == \{e_2, e_4\}$$

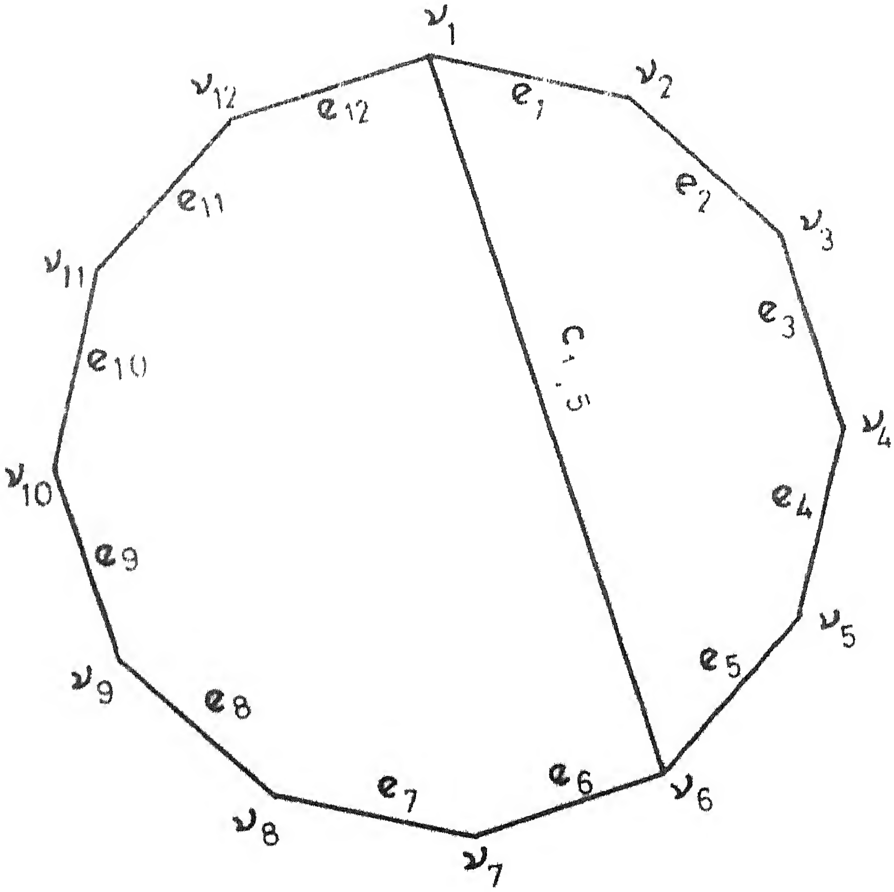
$$M_{15} = \{C_{1,5}\}$$

$$M_{16} = \{e_7, e_9, e_{11}\}$$

<u>Remark 3.3.1</u> Let G be an even cycle with chords. Let $C_{i,r}$ be a chord satisfying the condition of the Theorem 3.3.3. Suppose that G has a chord $C_{i,2t}$ which makes the cycle $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+2t}, v_i$. G may not be a B^*-graph in this case. But the fact that $C_{i,r}$ belongs to a maximum independent set still holds good because of (3.3.32), (3.3.33) and (3.3.34).

<u>COROLLARY 3.3.2</u> C_{2n} is a B^*-graph.

The next result is aimed at improving the result of Theorem 3.3.3.



v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8
v_9
v_{10}
v_{11}
v_{12}
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9
e_{10}
e_{11}
e_{12}
$c_{1,5}$

Fig. 3.3.1

LEMMA 3.3.5 Suppose that G is an even cycle with chords. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of G. Let $e_i = (v_i, v_{i+1})$, $1 \leq i \leq 2n$ and $C_{i,r} = (v_i, v_{i+r})$, $2 \leq r \leq 2n-2$, (suffix modulo 2n) be the edges and chords of G respectively. If corresponding to each chord $C_{i,r}$ which makes a cycle $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+r}, v_i$ that is isomorphic to C_{2t+1} there exists another chord $C_{i+1,r}$, then G is a B^*-graph.

Proof : By the hypothesis r equals 2t.

We shall prove that $C_{i,2t}$ and $C_{i+1,2t}$ belong to a maximum independent set. The sets

$$M_{17} = \{C_{i,2t}, C_{i+1,2t}\} \tag{3.3.35}$$

$$M_{18} = \{e_{i+2}, e_{i+4}, e_{i+6}, \ldots, e_{i+2t-2}\} \tag{3.3.36}$$

$$M_{19} = \{e_{i+2t+2}, e_{i+2t+4}, e_{i+2t+6}, \ldots, e_{i+2n-2}\} \tag{3.3.37}$$

are disjoint independent sets. The orders of these sets are 2, t-1 and n-t-1 respectively. The number of elements in the union of M_{17}, M_{18} and M_{19} is n. Hence the chords $C_{i,2t}$ and $C_{i+1,2t}$ are in a maximum independent set. The fact that each e_i belongs to a maximum independent set can be proved as in Theorem 3.3.3. //

Figure 3.3.2 illustrates Lemma 3.3.5.

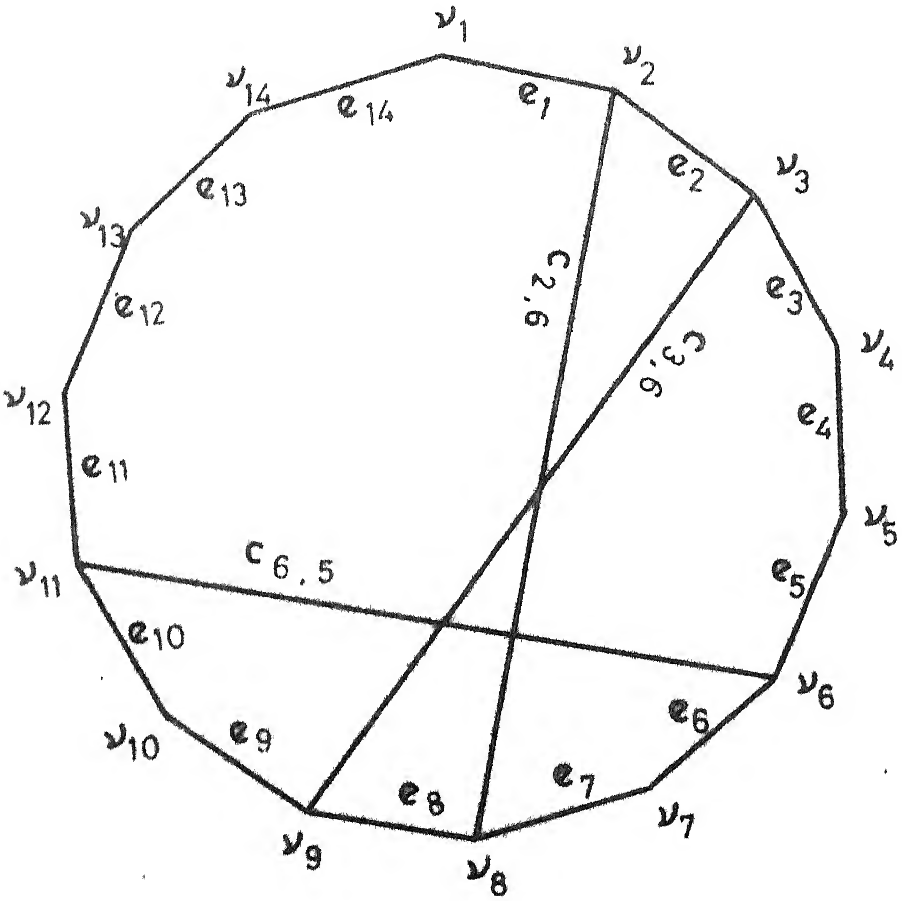

ν_1
ν_2
ν_3
ν_4
ν_5
ν_6
ν_7
ν_8
ν_9
ν_{10}
ν_{11}
ν_{12}
ν_{13}
ν_{14}
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9
e_{10}
e_{11}
e_{12}
e_{13}
e_{14}
$c_{2,6}$
$c_{3,6}$
$c_{6,5}$

Fig. 3.3.2

For this graph,

$$M_{17} = \{C_{2,6}, C_{3,6}\}$$

$$M_{18} = \{e_4, e_6\}$$

$$M_{19} = \{e_{10}, e_{12}, e_{14}\}$$

<u>THEOREM 3.3.4</u> Powers of cycles are B^*-graphs.

<u>Proof</u> : From Theorem 3.3.2 we get powers of odd cycles to be B^*-graphs since

$$C^r_{2n+1}, \ (2 \leq r \leq n) \tag{3.3.38}$$

is an odd cycle with chords.

Consider the r^{th} $(2 \leq r \leq n)$ power of an even cycle. Any chord in

$$C^r_{2n}, \ (2 \leq r \leq n) \tag{3.3.39}$$

belongs to one of the following types :

i) $C_{i,2t}$ (3.3.40)

ii) $C_{i,2t+1}$ (3.3.41)

If the chord $C_{i,2t}$ is present in C^r_{2n}, then $C_{i+1,2t}$ also exists. Hence by Lemma 3.3.5 C^r_{2n} is a B^*-graph. This implies that $C_{i,2t}$ is in a maximum independent set. In the remaining case by the Remark 3.3.1, we see that $C_{i,2t+1}$ is in a maximum independent set. Therefore powers of even cycles are also B^*-graphs.

COROLLARY 3.3.3 $L(C_n^r)$, $(1 \leq r \leq [n/2])$ is a B-graph.

Proof : By the definition of the line graph of a graph, since each edge of C_n^r is in a maximum independent set of edges, each vertex of $L(C_n^r)$ is in a MISS of $L(C_n^r)$. //

The next result suggests that we need not look for B^*-graphs among acyclic graphs which have perfect matching.

LEMMA 3.3.6 If an acyclic graph has a perfect matching, then it cannot be a B*-graph.

Proof : Let M be the unique perfect matching of an acyclic graph G with p vertices. Then

$$M \text{ has } p/2 \text{ edges} \tag{3.3.42}$$

$$G \text{ has } p-1 \text{ edges.} \tag{3.3.43}$$

Since the matching M is unique, there is no perfect matching involving the remaining $\frac{p}{2} - 1$ edges. Hence these edges are not in any maximum independent set. So G is not a B^*-graph. //

THEOREM 3.3.5 $L(C_n^r)$, $(1 \leq r \leq [n/2])$ is a B^*-graph.

Proof : We shall consider the case $r = \frac{n}{2}$ separately. Let $v_1, v_2, \ldots, v_n, v_1$ be the cycle C_n. Each vertex of C_n^r $(1 \leq r < \frac{n}{2})$ is adjacent to 2r vertices. Hence C_n^r has nr edges. Therefore $L(C_n^r)$ is of order nr.

Next we prove that $L(C_n^r)$ has a spanning cycle. Let us label the edges of C_n^r by $e_1, e_2, e_3, \ldots, e_{nr}$ starting from the

edge (v_n, v_1) and proceeding in the order given below.

$(v_n,v_1)(v_{n-1},v_1)(v_{n-2},v_1) \ldots (v_{n-(r-1)},v_1)(v_1,v_{r+1})(v_1,v_r)$

$(v_1,v_{r-1}) \ldots (v_1,v_2)(v_n,v_2)(v_{n-1},v_2) \ldots (v_{n-(r-2)}, v_2)$

$(v_2,v_{r+2})(v_2,v_{r+1})(v_2,v_r) \ldots (v_2,v_3)(v_n,v_3)(v_{n-1},v_3) \ldots$

$(v_{n-(r-3)}, v_3)(v_3,v_{r+3})(v_3,v_{r+2}) \ldots (v_{n-1}, v_n)$.

This ordering of the edges is illustrated in the Figure 3.3.3 for ${C_{10}}^2$. As a result,

$e_1, e_2, e_3, \ldots, e_{nr}, e_1$ is a spanning cycle of $L(C_n^r)$, $(1 \leq r < \frac{n}{2})$.

Each edge in the line graph of C_n^r arises from two adjacent edges of C_n^r. Consider the following sets of edges of $L(C_n^r)$

$$M_{20} = \{(e_1,e_2),(e_3,e_4),(e_5,e_6), \ldots, (e_{nr-1},e_{nr})\} \quad (3.3.44)$$

$$M_{21} = \{(e_1,e_2),(e_3,e_4),(e_5,e_6), \ldots, (e_{nr-2},e_{nr-1})\} \quad (3.3.45)$$

M_{20} and M_{21} are independent sets since no vertex e_i is repeated. If nr is even M_{20} contains nr/2 independent edges. Therefore it is a maximum independent set. If nr is odd M_{21} has [nr/2] edges and is a maximum independent set.

Now we shall prove that each edge of $L(C_n^r)$ arising out of any two adjacent edges incident with the vertex v_1 is in a maximum independent set. The 2r edges incident with v_1

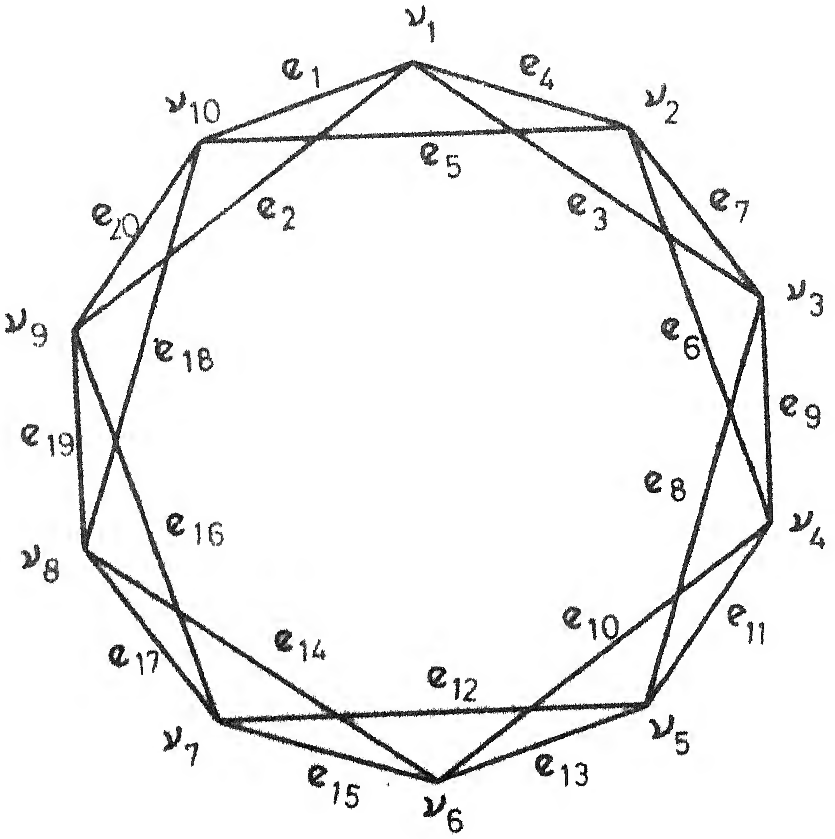

v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8
v_9
v_{10}
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9
e_{10}
e_{11}
e_{12}
e_{13}
e_{14}
e_{15}
e_{16}
e_{17}
e_{18}
e_{19}
e_{20}

Fig. 3.3.3

correspond to K_{2r} in the line graph. We know from Theorem 3.3.4 that K_{2r} is a B*-graph. This implies that each edge of $L(C_n^r)$ which corresponds to two adjacent edges incident with v_1 in C_n^r is in an independent set M_{22} having r edges in the complete subgraph formed by $e_1, e_2, e_3, \ldots, e_{2r}$ in $L(C_n^r)$. Let

$$M_{23} = \{(e_1,e_2), (e_3,e_4), (e_5,e_6), \ldots, (e_{2r-1},e_{2r})\} \quad (3.3.46)$$

Now the result follows since $(M_{20} - M_{23}) \cup M_{22}$ or $(M_{21} - M_{23}) \cup M_{22}$ is an independent set containing $[nr/2]$ edges.

When $r = n/2$, C_n^r becomes the complete graph K_{2r}. It has $r(2r-1)$ edges. So $L(K_{2r})$ is of order $r(2r-1)$. Labelling the edges as in the former case, we get $e_1, e_2, e_3, \ldots, e_{r(2r-1)}, e_1$ to be a spanning cycle of $L(K_{2r})$. If r is odd, then $L(K_{2r})$ is an odd cycle with chords and hence it is a B*-graph by Theorem 3.3.2.

Suppose that r is even. Let $e_1, e_2, e_3, \ldots, e_{2r-1}$ be the edges of K_{2r} incident with v_1. By an argument similar to that used in the former case we can prove that all the edges of $L(K_{2r})$ arising out of the adjacencies of $\{e_1, e_2, \ldots, e_{2r-2}\}$ belong to a maximum independent set. Similarly for $\{e_2, e_3, e_4, \ldots, e_{2r-1}\}$. We shall prove that (e_1, e_{2r-1}) also belongs to a maximum independent set. Consider the set

$$M_{24} = \{(e_1,e_{2r-1}),(e_{2r-2},e_{2r-3}), \ldots, (e_4,e_3)(e_2,e_{r(2r-1)}) \\ (e_{r(2r-1)-1},e_{r(2r-1)-2}),(e_{r(2r-1)-3},e_{r(2r-1)-4}) \ldots \\ (e_{2r+1},e_{2r})\} \quad (3.3.47)$$

This is a maximum independent set containing the edge (e_1, e_{2r-1}). //

3.4 B^{**}-graphs

In this section we discuss B^{**}-graphs and identify a class of graphs which cannot be B^{**}-graphs.

From Theorems 3.2.1 and 3.3.1 we see that P_{2n} is a B-graph, but it is not a B^*-graph. P_{2n+1} is a B^*-graph, but it is not a B-graph. Hence we conclude that a path cannot be a B^{**}-graph. A more general result is given in the following Theorem.

THEOREM 3.4.1 An acyclic graph without isolated vertices, cannot be a B^{**}-graph.

Proof : From Theorem 3.2.3 we get that an acyclic graph without isolated vertices is a B-graph if and only if it has a perfect matching. Lemma 3.3.6 states that if an acyclic graph has a perfect matching it cannot be a B^*-graph. Combining these two we see that no acyclic graph without isolated vertices is a B^{**}-graph. //

THEOREM 3.4.2 Cycles and their powers are B^{**}-graphs.

Proof : Combining Theorem 3.2.2 and Corollaries 3.3.1 and 3.3.2 we see that cycles are B^{**}-graphs. From Theorems 3.2.4 and 3.3.4 it follows that the powers of cycles are B^{**}-graphs.//

<u>THEOREM 3.4.3</u> $L(C_n^r)$ is a B^{**}-graph.

<u>Proof</u> : From Corollary 3.3.3 and Theorem 3.3.5 we observe that $L(C_n^r)$ is a B-graph as well as a B^*-graph. //

<u>COROLLARY 3.4.1</u> $T(K_n)$ is a B^{**}-graph.

<u>Proof</u> : $L(K_{2p})$ and $L(K_{2p+1})$ are B^{**}-graphs by Theorem 3.4.3. $L(K_{r+1})$ is isomorphic to $T(K_r)$ [5]. //

CHAPTER 4

ON B-POINT CRITICAL GRAPHS

CHAPTER 4

On B-point Critical Graphs

4.1 Introduction

Graphs critical with respect to some property have great fascination for graph theorists. One reason for this is that all critical graphs possess more structural properties than the ordinary graphs. In the third chapter we have studied B-graphs. A good characterization of B-graphs seems to be difficult. Here we concentrate on B-point critical graphs. Ravindra [103] has studied these graphs. He has proposed two conjectures.

C4.1.1 G is B-point critical implies that G is decomposable. (4.1.1)

C4.1.2 G is B-point critical implies that $\theta(G) = \alpha(G)$ (4.1.2)

It was claimed that C4.1.1 is weaker than C4.1.2. In this chapter we prove that (4.1.2) is not true by producing various counter examples. A few B-point critical graphs are also discussed.

We use the following results of [59].

If V_k's $(1 \leq k \leq r,\ r > 1)$ partition the vertices of a graph G and G_k is the induced subgraph of G on V_k, then

$$\alpha_o(G) \geq \sum_{k=1}^{r} \alpha_o(G_k) \qquad (4.1.3)$$

Using (3.2.23) we get

$$\alpha(G) \leq \sum_{k=1}^{r} \alpha(G_k) \qquad (4.1.4)$$

If $\alpha_o(G) = \sum_{k=1}^{r} \alpha_o(G_k)$, then G is decomposable

(4.1.5)

4.2 B-point critical graphs

We prove here that C_{2n}, C_{2n}^{n-1} $(n \geq 2)$, C_{nr}^{n-1}, $n,r \geq 2$ and join of two B-graphs with the same independence number are some classes of B-point critical graphs.

THEOREM 4.2.1 C_{2n} is a B-point critical graph.

Proof : We know from Theorem 3.2.2 that C_{2n} is a B-graph. $C_{2n}-v$, for every $v \varepsilon V$, is P_{2n-1}. Hence by Theorem 3.2.1 it is not a B-graph. So C_{2n} is a B-point critical graph. //

Our next Theorem extends this result to the $(r-1)^{th}$ power of an even cycle.

THEOREM 4.2.2 C_{2r}^{r-1} $(r \geq 2)$ is B-point critical.

Proof : Let $v_1, v_2, v_3, \ldots, v_{2r}, v_1$ be the cycle C_{2r}. Any vertex v_i $(1 \leq i \leq 2r)$ of C_{2r}^{r-1} is adjacent to all the other vertices except v_{i+r} (suffix modulo 2r). $\{v_i, v_{i+r}\}$ is a MISS. Each vertex v_i is in a MISS. Hence C_{2r}^{r-1} is a B-graph.

Next we shall prove that $C_{2r}^{r-1} - v_i$ is not a B-graph. By definition, the vertex v_{i+r} is adjacent to all the vertices of $C_{2r}^{r-1} - v_i$. $\{v_j, v_{j+r}\}$ $(j \neq i)$ is a MISS in $C_{2r}^{r-1} - v_i$.

Therefore v_{i+r} is not in any MISS of $C_{2r}^{r-1} - v_i$. So C_{2r}^{r-1} satisfies all the conditions for a B-point critical graph. //

The next Theorem gives a necessary and sufficient condition for the join of two B-point critical graphs to be B-point critical.

THEOREM 4.2.3 If G_1 and G_2 are B-point critical, then G_1+G_2 is B-point critical if and only if $\alpha(G_1) = \alpha(G_2)$.

Proof : We have, from Theorem 3.2.5, that if G_1 and G_2 are B-graphs and $\alpha(G_1) = \alpha(G_2)$, then G_1+G_2 is a B-graph. By hypothesis G_1 is a B-point critical graph. G_1-v, for every $v \epsilon V(G_1)$ is not a B-graph. There exists at least one vertex, say v_1 which is not in any MISS of G_1-v. In G_1+G_2, v_1 is adjacent to every vertex of G_2. As a result, v_1 cannot be in any MISS of $G_1+G_2 - v$. This is true of vertices of G_2 also. Hence G_1+G_2-v, for every $v \epsilon V (G_1+G_2)$, is not a B-graph.

The second part of the proof is simple. If $\alpha(G_1) \neq \alpha(G_2)$, then we can prove that G_1+G_2 is not even a B-graph. Without loss of generality we can assume that

$$\alpha(G_1) < \alpha(G_2) \tag{4.2.1}$$

In G_1+G_2, each vertex of G_1 is adjacent to all the vertices of G_2. Therefore no vertex of G_1 can belong to an independent set consisting of $\alpha(G_2)$ vertices. So G_1+G_2 is not a B-graph. //

This Theorem helps us to construct new B-point critical graphs from known ones. For example

$$C_{2r},\ C_{2r}+C_{2r},\ C_{2r}+C_{2r}+C_{2r},\ \ldots, \tag{4.2.2}$$

$$C_{2r}^{r-1},\ C_{2r}^{r-1}+C_{2r}^{r-1},\ C_{2r}^{r-1}+C_{2r}^{r-1}+C_{2r}^{r-1},\ \ldots \tag{4.2.3}$$

are B-point critical graphs.

A Theorem which is more general than Theorem 4.2.2 is proved next.

THEOREM 4.2.4 C_{nr}^{n-1}, $n,r \geq 2$ is B-point critical.

Proof : Let $v_1, v_2, v_3, \ldots, v_{nr}, v_1$ be the cycle C_{nr}. C_{nr}^{n-1} is a B-graph by Theorem 3.2.4. Each vertex v_i $(1 \leq i \leq nr)$ of C_{nr}^{n-1} is adjacent to

$$S_1 = \{v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{i+n-1}\} \tag{4.2.4}$$

and

$$S_2 = \{v_{i+nr-1}, v_{i+nr-2}, \ldots, v_{i+nr-(n-1)}\} \tag{4.2.5}$$

(suffix modulo nr). We define

$$V_k = \{v_{(k-1)n+1}, v_{(k-1)n+2}, v_{(k-1)n+3}, \ldots, v_{kn}\} (1 \leq k \leq r) \tag{4.2.6}$$

$$G_k = \langle V_k \rangle \tag{4.2.7}$$

and

$$S_t = \{v_t, v_{n+t}, v_{2n+t}, \ldots, v_{jn+t}, \ldots, v_{(r-1)n+t}\} (1 \leq t \leq n) \tag{4.2.8}$$

Each set of n consecutive vertices of C_{nr}^{n-1} forms a clique of order n by (4.2.4). Hence

$$G_k = K_n \tag{4.2.9}$$

and $$\alpha(C_{nr}^{n-1}) \leq r \tag{4.2.10}$$

by (4.1.4) since $\alpha(K_n) = 1$. By definition, the set S_t is an independent set containing r vertices. Using (4.2.10) we see that S_t is a MISS and so

$$\alpha(C_{nr}^{n-1}) = r. \tag{4.2.11}$$

The vertex v_{t+jn} $(o \leq j \leq r-1)$ belongs to a MISS. If

$$t_1+j_1n = t_2+j_2n \qquad \begin{array}{l} 1 \leq t_1,\ t_2 \leq n \\ 1 \leq j_1, j_2 \leq r-1 \end{array} \tag{4.2.12}$$

Then $$t_1-t_2 = (j_2-j_1)n\ . \tag{4.2.13}$$

The R.H.S. is a multiple of n. But the L.H.S. is strictly less than n because of (4.2.12). Therefore equation (4.2.13) leads to a contradiction. Hence each vertex is in exactly one MISS S_t. As a result, in $C_{nr}^{n-1} - v_t$ the vertex v_{t+jn} $(1 \leq j \leq r-1)$ is not included in any MISS. So C_{nr}^{n-1} $(n,\ r \geq 2)$ is a B-point critical graph. //

Theorem 4.2.2 becomes a particular case of Theorem 4.2.4 when r = 2 and Theorem 4.2.1 is obtained if we replace n of Theorem 4.2.4 by 2.

4.3 Counter examples

In this section we present counter examples to (4.1.2). The simplest counter example is given in Figure 4.3.1.

4.3.1 A simple counter example to C4.1.2

Consider the graph with ten vertices as given in Figure 4.3.1. Each vertex v_i $(1 \leq i \leq 10)$ is adjacent to v_{i+2}, v_{i+3}, v_{i+7} and v_{i+8} (suffix modulo 10). We define

$$V_1 = \{ v_1, v_2, v_3, v_4, v_5 \} \tag{4.3.1}$$

$$V_2 = \{ v_6, v_7, v_8, v_9, v_{10} \} \tag{4.3.2}$$

$$G_1 = < V_1 > \tag{4.3.3}$$

$$G_2 = < V_2 > \tag{4.3.4}$$

and
$$S_t = \{ v_t, v_{t+1}, v_{t+5}, v_{t+6} \} \; (1 \leq t \leq 10) \tag{4.3.5}$$

Each vertex of V_1 is adjacent to exactly two of its vertices and $v_1, v_3, v_5, v_2, v_4, v_1$ is a cycle. Similarly for V_2. Hence G_i $(i = 1,2)$ is C_5. Using (4.1.4)

$$\alpha(G) \leq \alpha(G_1) + \alpha(G_2) \leq 4 \tag{4.3.6}$$

since $\alpha(C_5) = 2$.

S_t is an independent set containing 4 vertices. So S_t is a MISS and

$$\alpha(G) = 4 \; . \tag{4.3.7}$$

The vertex v_t is in S_t. Therefore each vertex of G is in a MISS. That is G is a B-graph.

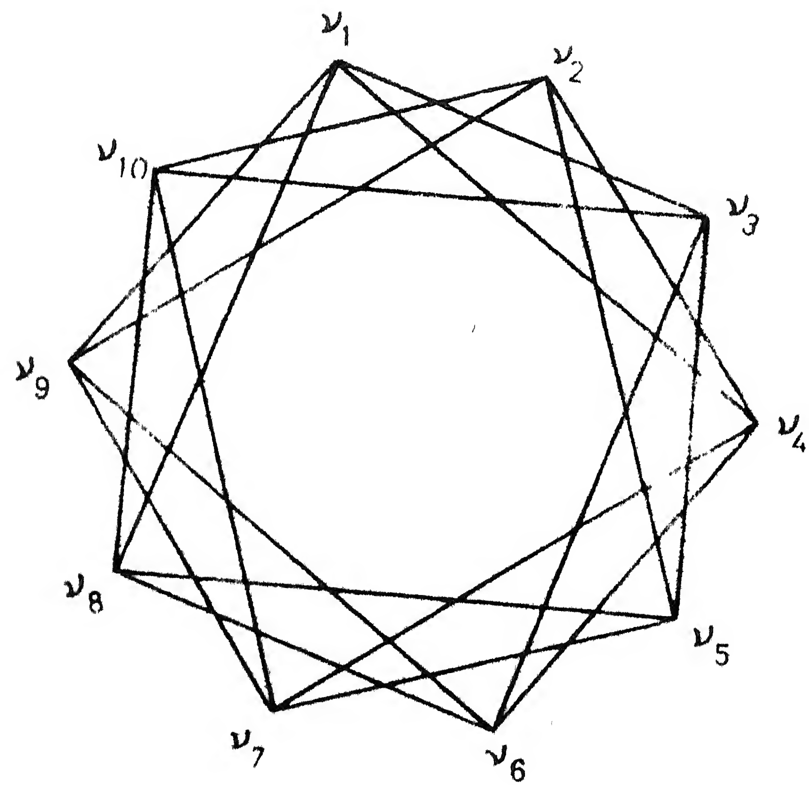


Fig. 4.3.1

Each MISS containing v_i must include either v_{i+1} or v_{i+9}. Suppose that there is a MISS S containing v_i and not having v_{i+1} and v_{i+9}. v_i is adjacent to v_{i+2} and v_{i+8}. The remaining vertices form a C_5. $\alpha(C_5) = 2$. Hence S contains at the most 3 vertices. This contradicts the fact that S is a MISS. As a result, each MISS contains 2 sets of 2 consecutive vertices of G. Each vertex is in exactly two MISSes. The vertices v_t and v_{t+5} appear in S_t and S_{t+4} only. Hence in $G-v_t$, v_{t+5} is not included in any MISS. Therefore $G-v_t$ $(1 \leq t \leq 10)$ is not a B-graph. Thus G satisfies all the conditions for a B-point critical graph.

Next we shall find the covering number of G. The vertex v_i is adjacent to v_{i+2}, v_{i+3}, v_{i+7} and v_{i+8}.

$$S_{i+2} = \{v_{i+2}, v_{i+3}, v_{i+7}, v_{i+8}\} \qquad (4.3.8)$$

Equation (4.3.8) can be obtained from (4.3.5) when t is replaced by i+2. Hence each vertex of G is adjacent to vertices forming an independent set. This implies that G is a triangle-free graph. Therefore the minimum number of cliques needed to cover all the vertices of G ≥ 5.

$$\theta(G) \geq 5 . \qquad (4.3.9)$$

But the set

$$M = \{(v_1,v_3),(v_2,v_4),(v_5,v_7),(v_6,v_9),(v_8,v_{10})\} \qquad (4.3.10)$$

covers all the vertices of G. Combining (4.3.9) and (4.3.10) we get

$$\theta(G) = 5 . \tag{4.3.11}$$

Now from (4.3.7) and (4.3.11) we see that G is a B-point critical graph for which $\alpha(G) < \theta(G)$.

This counter example has been accepted for publication.

For the counter example given here $\theta = \alpha+1$. It is possible to generalize the graph presented in Figure 4.3.1. We produce a B-point critical graph with the property that $\theta = \alpha+n$, where n is any positive integer.

4.3.2 A generalized counter example to C4.1.2

Consider a graph G with 5r $(r \geq 2)$ vertices labelled $v_1, v_2, v_3, \ldots, v_{5r}$. Any vertex v_i $(1 \leq i \leq 5r)$ is adjacent to the vertices of

$$S = \{v_{i+2}, v_{i+3}, v_{i+5r-2}, v_{i+5r-3}\} \tag{4.3.12}$$

(suffix modulo 5r). We define S_{i+t} $(o \leq t \leq 4)$ as

$$S_{i+t} = \bigcup_{j=o}^{r-1} S^j_{i+t} \tag{4.3.13}$$

$$S^j_{i+t} = \{v_{i+5j+t}, v_{i+5j+1+t}\} \; (o \leq j \leq r-1) \tag{4.3.14}$$

$$V_k = \{v_{5(k-1)+1}, v_{5(k-1)+2}, \ldots, v_{5(k-1)+m}, \ldots, v_{5k}\} \quad (1 \leq k \leq r) \tag{4.3.15}$$

$$G_k = \langle V_k \rangle \tag{4.3.16}$$

From the adjacency defined in (4.3.12) we observe the following. If we consider any 5 consecutive vertices, say $v_i, v_{i+1}, v_{i+2}, v_{i+3}$

and v_{i+4} of G, each vertex is adjacent to exactly two vertices of this set. And $v_i, v_{i+3}, v_{i+1}, v_{i+4}, v_{i+2}, v_i$ is a cycle. Hence

$$G_k = C_5 \quad (1 \leq k \leq r) \tag{4.3.17}$$

$$\alpha(G) \leq \sum_{k=1}^{r} \alpha(G_k) \leq 2r \tag{4.3.18}$$

by (4.1.4) since $\alpha(C_5) = 2$. By definition, S_{i+t} is an independent set containing 2r elements. Therefore S_{i+t} is a MISS. So

$$\alpha(G) = 2r . \tag{4.3.19}$$

Each vertex v_i is in a MISS of G.

$$G \text{ is a B-graph.} \tag{4.3.20}$$

Every MISS containing v_i must contain either v_{i+1} or v_{i+5r-1}. Suppose that there exists a MISS S' containing v_i and not containing v_{i+1} and v_{i+5r-1}. The vertex v_i is adjacent to v_{i+2} and v_{i+5r-2}. As a result, we are left with 5(r-1) consecutive vertices of G in order to form an independent set containing v_i. We can select only 2(r-1) independent vertices since each set of 5 consecutive vertices forms a C_5. This contradicts the fact that S' is a MISS. Hence each MISS contains r sets of 2 consecutive vertices and each vertex appears in exactly two MISSes. The vertices $v_i, v_{i+5}, v_{i+10}, \ldots, v_{i+5(r-1)}$ appear only in the MISSes S_i and S_{i+4}. In $G-v_i$, the vertex v_{i+5n} $(1 \leq n \leq r-1)$ is not present in any MISS. That means

$$G-v_i \ (1 \leq i \leq 5r) \text{ is not a B-graph.} \tag{4.3.21}$$

From (4.3.20) and (4.3.21) we get that G is a B-point critical graph.

Now we find the covering number of G. The vertex v_i is adjacent to the vertices of S only.

$$S \subset S_{i+2} \tag{4.3.22}$$

Each vertex is adjacent to vertices forming an independent set. This implies that

$$\omega(G) = 2 \ . \tag{4.3.23}$$

Therefore a minimum covering of G would require at least $\{\frac{5r}{2}\}$ edges. In the following 4 cases m stands for a positive integer.

Case 1 $5r = 4m$.

G is a triangle-free graph by (4.3.23). A covering of G using 2m edges is a minimum covering.

$$M = \bigcup_{t=o}^{m-1} \{(v_{1+4t}, v_{1+4t+2}), (v_{1+4t+1}, v_{1+4t+3})\} \tag{4.3.24}$$

covers all the vertices of G. That means

$$\theta(G) = \frac{5r}{2} \tag{4.3.25}$$

So
$$\theta(G) = \alpha(G) + \frac{r}{2} \ . \tag{4.3.26}$$

Case 2 $5r = 4m+1$.

In this case M and the trivial clique $\{v_{5r}\}$ cover all the vertices of G.

$$\theta(G) = 2m + 1 = \frac{5r+1}{2} \ . \tag{4.3.27}$$

Therefore $\theta(G) = \alpha(G) + \frac{r+1}{2}$. (4.3.28)

Case 3 $5r = 4m+2$.

The set

$$M_1 = \bigcup_{t=0}^{m-2} \{(v_{1+4t}, v_{1+4t+2}), (v_{1+4t+1}, v_{1+4t+3})\}$$

and the edges $(v_{5r-5}, v_{5r-2}), (v_{5r-4}, v_{5r-1}), (v_{5r-3}, v_{5r})$ cover all the vertices of G.

$$\theta(G) = 2m+1 = \frac{5r}{2} \qquad (4.3.29)$$

Hence $\theta(G) = \alpha(G) + \frac{r}{2}$. (4.3.30)

Case 4 $5r = 4m+3$.

In the present case M, the edge (v_{5r-2}, v_{5r}) and the trivial clique $\{v_{5r-1}\}$ cover the vertices of G.

$$\theta(G) = \frac{5r+1}{2} \qquad (4.3.31)$$

That means $\theta(G) = \alpha(G) + \frac{r+1}{2}$. (4.3.32)

From these 4 cases it is easy to see that for each positive integer n it is possible to produce a B-point critical graph G with $\theta = \alpha+n$.

Figure 4.3.2 illustrates the counter example for the value $r = 3$.

The counter example presented in Figure 4.3.1 is a particular case of G when $r = 2$. It is also possible to construct a much more general counter example to the conjecture C4.1.2. We discuss it in the next section.

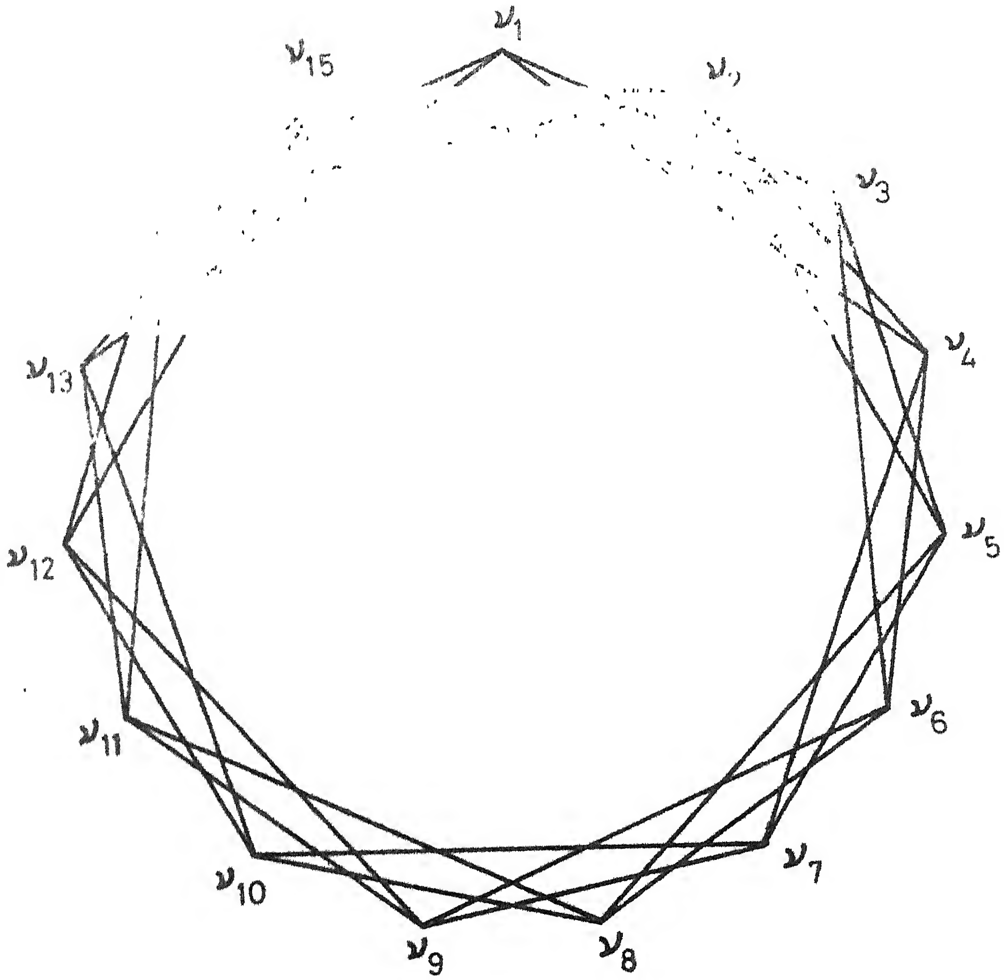


Fig. 4.3.2

4.3.3 A generalization of the counter example in section 4.3.2

G is a graph with $r(2n+1)(r,n \geq 2)$ vertices labelled, $v_1, v_2, v_3, \ldots, v_{r(2n+1)}$. Any vertex v_i $(1 \leq i \leq r(2n+1))$ is adjacent to the vertices of

$$S = \{v_{i+n}, v_{i+n+1}, v_{i+r(2n+1)-n}, v_{i+r(2n+1)-(n+1)}\} \tag{4.3.33}$$

(suffix modulo $r(2n+1)$). We define

S_{i+t} $(o \leq t \leq 2n)$ as

$$S_{i+t} = \bigcup_{j=o}^{r-1} S^j_{i+t} \tag{4.3.34}$$

$$S^j_{i+t} = \{v_{i+j(2n+1)+t}, v_{i+j(2n+1)+t+1}, v_{i+j(2n+1)+t+2}, \ldots, v_{i+j(2n+1)+t+m}, \ldots, v_{i+j(2n+1)+t+n-1}\} \tag{4.3.35}$$

$$V_k = \{v_{(k-1)(2n+1)+1}, v_{(k-1)(2n+1)+2}, v_{(k-1)(2n+1)+3}, \ldots, v_{(k-1)(2n+1)+m}, \ldots, v_{k(2n+1)}\} \quad (1 \leq k \leq r) \tag{4.3.36}$$

$$G_k = \langle V_k \rangle \tag{4.3.37}$$

Consider $(2n+1)$ consecutive vertices of G, say $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+2n}$. Each vertex of this set is adjacent to exactly two vertices of it. The sequence $v_i, v_{i+n}, v_{i+2n}, v_{i+n-1}, v_{i+2n-1}, v_{i+n-2}, v_{i+2n-2}, \ldots, v_{i+n-s}, v_{i+2n-s}, \ldots, v_{i+1}, v_{i+n+1}, v_i$ forms a cycle. Therefore,

$$G_k = C_{2n+1} \tag{4.3.38}$$

$$\alpha(G) \leq \sum_{k=1}^{r} \alpha(G_k) \leq nr \tag{4.3.39}$$

by (4.1.4) since $\alpha(C_{2n+1}) = n$.

By definition, S_{i+t} is an independent set containing nr vertices. Hence S_{i+t} is a MISS and

$$\alpha(G) = nr \; . \tag{4.3.40}$$

Each vertex v_i is in a MISS of G.

$$G \text{ is a B-graph.} \tag{4.3.41}$$

Every MISS containing v_i must also contain n consecutive vertices of G including v_i by definition. Each MISS contains r sets of n consecutive vertices. As a result, each vertex appears in n and only n MISSes. The vertices $v_i, v_{i+2n+1}, v_{i+2(2n+1)}, v_{i+3(2n+1)}, \ldots, v_{i+(r-1)(2n+1)}$ appear in the MISSes $S_i, S_{i+2n}, S_{i+2n-1}, S_{i+2n-2}, S_{i+2n-3}, \ldots, S_{i+n+2}$ only. Therefore in $G-v_i, v_{i+j(2n+1)}$ $(1 \leq j \leq r-1)$ is not in any maximum independent set.

$$G-v_i \quad (1 \leq i \leq r(2n+1)) \text{ is not a B-graph.} \tag{4.3.42}$$

From (4.3.41) and (4.3.42) we get that G is a B-point critical graph.

Next we show that $\alpha(G) < \theta(G)$. The vertex v_i is adjacent to the vertices of S only.

$$S \subset S_{i+n} \tag{4.3.43}$$

This means that each vertex of G is adjacent to vertices forming an independent set only. So

$$G \text{ is a triangle-free graph.} \tag{4.3.44}$$

G has $r(2n+1)$ vertices. Therefore, the covering number of G $> nr+1$ because of (4.3.44). This implies that $\alpha(G) < \theta(G)$.

Figure 4.3.3 illustrates this counter example for the value $n = 3$ and $r = 2$.

The counter example given in section 4.3.1 is the graph G for which $n = r = 2$ and that given in section 4.3.2 is G with $n = 2$.

There is another possible generalization of the counter example of section 4.3.2. Apart from a counter example it also serves the purpose of getting another set of B-point critical graphs.

4.3.4 Another generalization of the counter example of section 4.3.1

G is a graph with $r(2n+1)$ vertices labelled $v_1, v_2, v_3, \ldots, v_{r(2n+1)}$. Any vertex v_i is adjacent to

$$S = \{v_{i+2}, v_{i+3}, v_{i+4}, \ldots, v_{i+2n-1}\} \tag{4.3.45}$$

and

$$S' = \{v_{i+r(2n+1)-2}, v_{i+r(2n+1)-3}, v_{i+r(2n+1)-4}, \ldots, v_{i+r(2n+1)-(2n-1)}\} \tag{4.3.46}$$

Let us define S_{i+t} $(o \leq t \leq 2n)$ as

$$S_{i+t} = \bigcup_{j=o}^{r-1} S_{i+t}^{j} \tag{4.3.47}$$

$$S_{i+t}^{j} = \{v_{i+j(2n+1)+t}, v_{i+j(2n+1)+t+1}\} \tag{4.3.48}$$

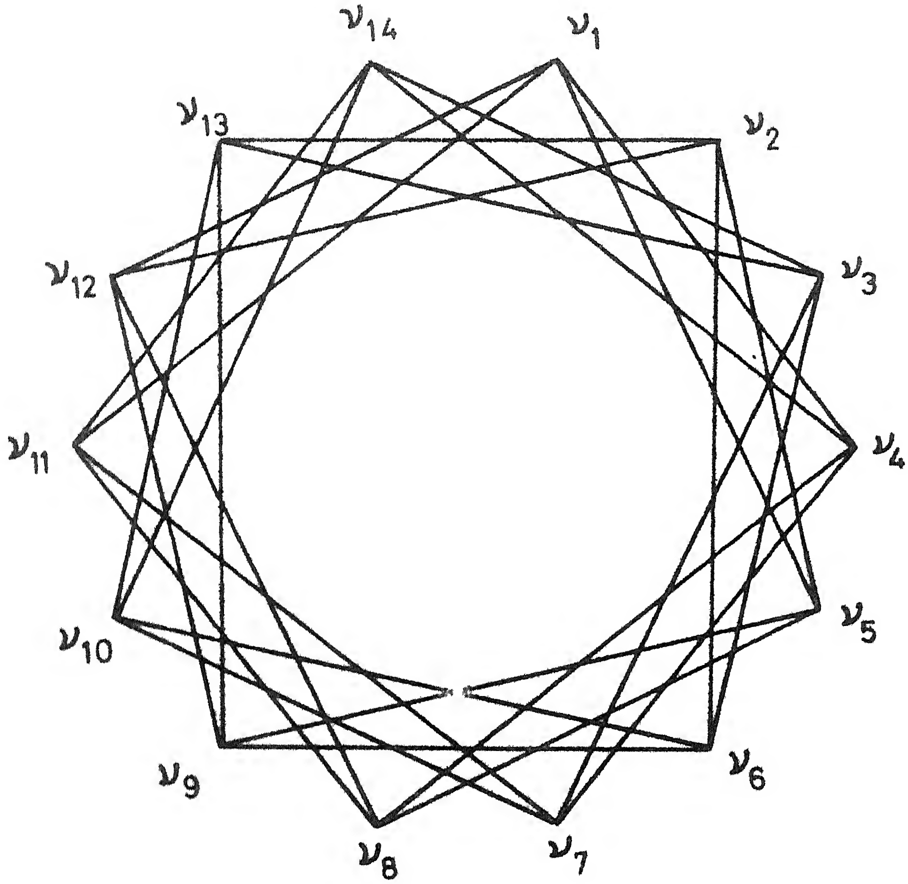


Fig. 4.3.3

$$V_k = \{ v_{(k-1)(2n+1)+1}, v_{(k-1)(2n+1)+2}, v_{(k-1)(2n+1)+3}, \ldots, v_{k(2n+1)} \} \quad (1 \le k \le r) \tag{4.3.49}$$

$$G_k = \langle V_k \rangle \tag{4.3.50}$$

Because of the adjacency (4.3.45) and (4.3.46), if we consider 2n+1 consecutive vertices of G, say $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+2n}$, then each vertex of this set is not adjacent to exactly two of its vertices. Hence in the complement of the graph formed by these vertices, each vertex is adjacent to exactly two vertices and $v_i, v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{i+2n}, v_i$ is a cycle. So that $\overline{G}_k = C_{2n+1}$.

$$G_k = \overline{C_{2n+1}} \,. \tag{4.3.51}$$

$$\alpha(G) \le \sum_{k=1}^{r} \alpha(G_k) \le 2r \tag{4.3.52}$$

by (4.1.4) since $\alpha(\overline{C_{2n+1}}) = 2$.

By definition, S_{i+t} is an independent set containing 2r vertices. Therefore S_{i+t} is a MISS and

$$\alpha(G) = 2r. \tag{4.3.53}$$

Each vertex of G is in a MISS of G.

$$G \text{ is a B-graph.} \tag{4.3.54}$$

Each MISS contains r sets of 2 consecutive vertices and each vertex is in exactly two MISSes. The vertices v_{i+2n+1}, $v_{i+2(2n+1)}, v_{i+3(2n+1)}, \ldots, v_{i+(r-1)(2n+1)}$ appear in both

the MISSes containing v_i namely S_i and S_{i+2n}. So in $G-v_i$, these vertices are not present in any MISS.

$G-v_i$ $(1 \leq i \leq r(2n+1))$ is not a B-graph. (4.3.55)

If we combine (4.3.54) and (4.3.55) we can see that G is a B-point critical graph.

Now we shall find the density of G. Since any vertex v_i is adjacent to vertices of S and S' only, $\omega(G)$ is the same as ω of the induced subgraph of G on $S \cup S' \cup \{v_i\}$. Consecutive vertices of S and S' form an independent set. Hence we require at least 2m-1 consecutive vertices in order to form a clique of order m. Each of the sets S and S' is of order 2n-2. So the largest order of a clique of $\langle S \rangle$ is n-1. Similarly for S'. A clique of order n+1 in $\langle S \cup S' \cup \{v_i\} \rangle$ corresponds to a clique of order n in $\langle S \cup S' \rangle$. Let a clique of order n be formed by s vertices of S and n-s vertices of S'. Then at least

$$2s-1+3+2(n-s)-1 = 2n+1$$

vertices are involved. This is possible only when a vertex v_i is adjacent to the vertex v_{i+2n}. This contradicts (4.3.45). Therefore

$$\omega(G) \leq n. \qquad (4.3.56)$$

But

$$\omega(G) \geq n \qquad (4.3.57)$$

since $\overline{C_{2n+1}}$ is an induced subgraph of G. Combining (4.3.56) and (4.3.57), we get

$$\omega(G) = n \ . \tag{4.3.58}$$

G has r(2n+1) vertices. So the minimum number of cliques needed to cover all the vertices of G $\geq$ 2r+1 because of (4.3.58). This implies that

$$\alpha(G) < \theta(G).$$

Figure 4.3.4 illustrates this counter example for the value n = 3 and r = 2.

It is interesting to note that all the counter examples produced to disprove the conjecture (4.1.2) involve the two known sets of critical graphs namely C_{2n+1} and $\overline{C_{2n+1}}$, n $\geq$ 2.

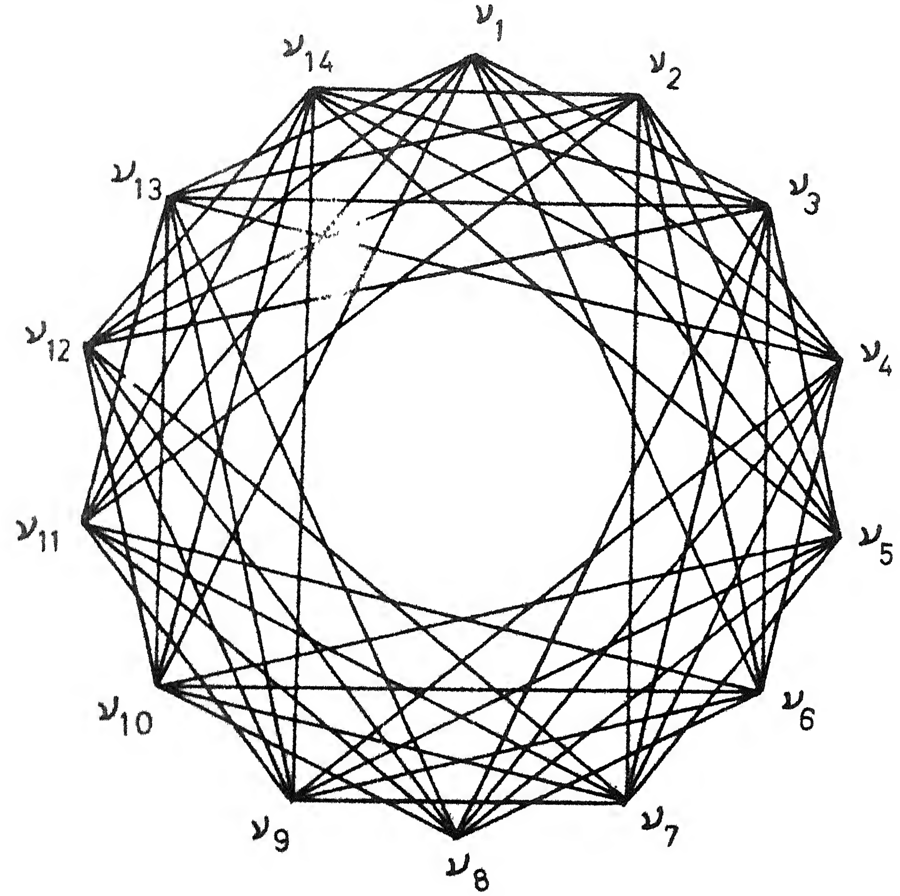


Fig. 4.3.4

CHAPTER 5

ON INDECOMPOSABLE GRAPHS

CHAPTER 5

On Indecomposable Graphs

5.1 Introduction

The concept of indecomposable graphs was introduced by Erdös and Gallai [36]. They present one of the important properties of indecomposable graphs namely if a nontrivial graph G with p points is indecomposable then $\alpha_o(G) \geq p/2$. Many authors have studied indecomposable graphs. Harary and Plummer [59] studied them in some detail. They constructed several infinite families of indecomposable graphs. Apart from finding some properties of indecomposable graphs they presented the following necessary condition for a graph to be decomposable. If G is a connected graph which is separated by the points of a complete subgraph K_n, then G is decomposable.

In this chapter we prove that all indecomposable graphs other than complete graphs are not perfect and that all critical graphs are indecomposable. It is shown that for an indecomposable graph G, $\alpha(G-Q) = \alpha$ for every clique Q of G. Some results are presented for indecomposable graphs with $\alpha = 2$, $\theta \geq 3$, powers of cycles, line graphs and total graphs. We use the following results of [59].

If $Cr(G)$ is a connected spanning subgraph of G, then G is indecomposable. (5.1.1)

If G is line critical, then G is indecomposable. (5.1.2)

If G is decomposable, then G has a decomposition $[G_1, G_2, \ldots, G_k]$ $(k > 1)$ where all the G_i's $(1 \leq i \leq k)$ are indecomposable [36]. (5.1.3)

5.2 Indecomposable and imperfect graphs

In this section we discuss some relationship between indecomposable and imperfect graphs. Complete graphs and odd cycles are indecomposable [59]. The following Theorem states that among all indecomposable graphs complete graphs alone belong to the class of perfect graphs.

THEOREM 5.2.1 Indecomposable graphs other than complete graphs are imperfect.

Proof : Suppose that an indecomposable graph G which is not a complete graph is perfect. Then

$$\alpha(H) = \theta(H) \tag{5.2.1}$$

for every induced subgraph H of G. In particular,

$$\alpha(G) = \theta(G) \tag{5.2.2}$$

This means that α cliques are enough to cover all the vertices of G. Let

$$S = [G_1, G_2, G_3, \ldots, G_\alpha] \tag{5.2.3}$$

be the cliques. Then

$$\alpha(G_i) = 1 \quad (1 \leq i \leq \alpha) \tag{5.2.4}$$

Hence $\alpha(G) = \sum_{i=1}^{\alpha} \alpha(G_i)$. (5.2.5)

From (5.2.5) we get that G is a decomposable graph and (5.2.3) gives a decomposition of G. This contradicts the fact that G is indecomposable. //

The converse of this result is not true. The graph in Figure 4.3.1 is one example. It is an imperfect graph since it contains C_5 as an induced subgraph.

$$\alpha(G) = 4$$

by (4.3.7). V_1 and V_2 partition V and

$$\alpha(G) = \alpha(G_1) + \alpha(G_2) \qquad (5.2.6)$$

since $G_1 = G_2 = C_5$.

Therefore $[G_1, G_2]$ is a decomposition of the imperfect graph G. That means an imperfect graph which is not a complete graph need not be indecomposable.

THEOREM 5.2.2 Critical graphs are indecomposable.

Proof : Suppose that G is a critical graph which is decomposable. Then there is a decomposition of G in terms of indecomposable graphs by (5.1.3), say

$$S = [G_1, G_2, G_3, \ldots, G_k] \quad (k > 1). \qquad (5.2.7)$$

Then $\alpha(G) = \sum_{i=1}^{k} \alpha(G_i)$. (5.2.8)

Since G is a critical graph, each proper induced subgraph of G is perfect. So for $1 \leq i \leq k$, $(k > 1)$, G_i is a perfect indecomposable graph. That is,

$$G_i (1 \leq i \leq k) \text{ is a complete graph }, \tag{5.2.9}$$

by Theorem 5.2.1. Therefore the graphs of (5.2.7) form a set of cliques which cover all the vertices of G. This implies that

$$\alpha(G) = \theta(G) \tag{5.2.10}$$

since

$$\alpha(H) = \theta(H) \tag{5.2.11}$$

for every induced subgraph H of G and

$$\alpha(G) = \theta(G),$$

G is α-perfect and also perfect [71]. This contradicts the fact that G is critical. //

In the next section we study some indecomposable graphs.

5.3 <u>Some Indecomposable Graphs</u>

The following Theorem gives a method to construct indecomposable graphs.

<u>THEOREM 5.3.1</u> A graph G with $\alpha = 2$ and $\theta \geq 3$ is indecomposable.

<u>Proof</u> : If possible, let a graph G with

$$\alpha = 2 \tag{5.3.1}$$

and

$$\theta \geq 3 \tag{5.3.2}$$

be decomposable. Because of (5.3.1) any decomposition of G

cannot contain more than two graphs. Let

$$S = [G_1, G_2] \qquad (5.3.3)$$

be a decomposition of G. Both G_1 and G_2 cannot be complete graphs. Otherwise

$$\theta(G) = 2, \qquad (5.3.4)$$

contradicting the hypothesis. Hence the minimum value of the sum of the independence numbers of G_1 and G_2 is 3. Therefore, G is indecomposable. //

The converse need not be true. For, complete graphs are indecomposable and have $\alpha = \theta = 1$.

COROLLARY 5.3.1 The complements of the odd cycles C_{2n+1}, $n \geq 2$ are indecomposable.

Proof follows by observing that

$$\omega(G) = \alpha(\bar{G}) \qquad (5.3.5)$$

$$\chi(G) = \theta(\bar{G}) \qquad (5.3.6)$$

[45], and for $n \geq 2$

$$\omega(C_{2n+1}) = 2 \qquad (5.3.7)$$

$$\chi(C_{2n+1}) = 3 \qquad (5.3.8)$$

[50]. //

Theorem 5.3.1 helps us to construct indecomposable graphs. Consider three disjoint cliques K_{n_1}, K_{n_2} and K_{n_3}. Let u be a vertex of K_{n_1} and v be a vertex of K_{n_2}. Remove the edge (u,v)

from the join of K_{n_1} and K_{n_2}. Let u be adjacent to n $(1 \leq n \leq n_3-1)$ vertices of K_{n_3} and v to the remaining n_3-n vertices of K_{n_3}. The resulting graph G has

$$\alpha = 2$$

and

$$\theta \geq 3 .$$

This G is indecomposable by Theorem 5.3.1.

Figure 5.3.1 illustrates this construction for the value $n_1 = 2$, $n_2 = 4$ and $n_3 = 3$.

The next Theorem states that some powers of cycles are indecomposable.

<u>THEOREM 5.3.2</u> C_{nr+k}^{n-1}, $n,r \geq 2$, $1 \leq k \leq n-1$ is indecomposable.

<u>Proof</u> : Let $v_1, v_2, v_3, \ldots, v_{nr+k}, v_1$ be the cycle C_{nr+k}. Any vertex v_i $(1 \leq i \leq nr+k)$ of C_{nr+k}^{n-1} is adjacent to the vertices of

$$S' = \{v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{i+n-1}\} \quad (5.3.9)$$

and

$$S'' = \{v_{i+nr+k-1}, v_{i+nr+k-2}, v_{i+nr+k-3}, \ldots, v_{i+nr-(n-k-1)}\} \quad (5.3.10)$$

(suffix modulo nr+k). The set

$$S = \{v_{i+1}, v_{i+n+1}, v_{i+2n+1}, v_{i+3n+1}, \ldots, v_{i+(r-1)n+1}\} \quad (5.3.11)$$

is an independent set with r vertices. Therefore,

$$\alpha(C_{nr+k}^{n-1}) \geq r . \quad (5.3.12)$$

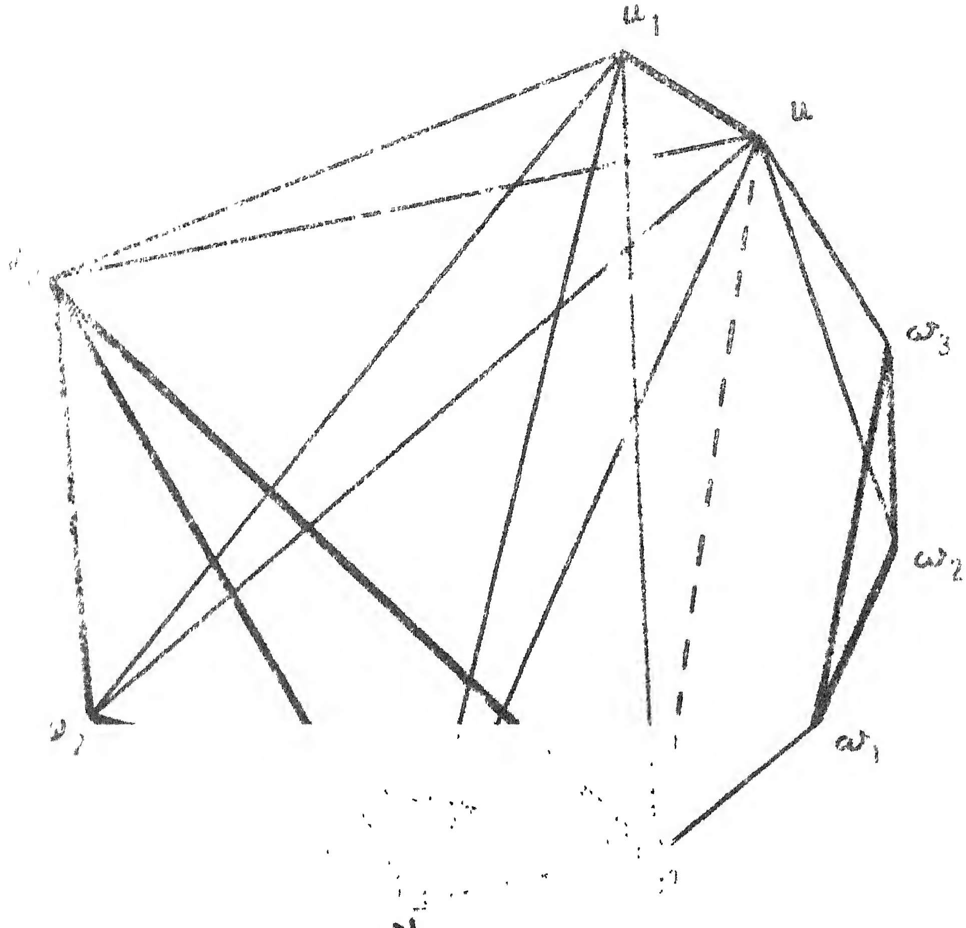


Fig. 5.3.1

Suppose that

$$\alpha(C^{n-1}_{nr+k}) = r+1 \ . \tag{5.3.13}$$

Then

$$\omega(\overline{C^{n-1}_{nr+k}}) = r+1 \ . \tag{5.3.14}$$

Let $u_1, u_2, u_3, \ldots, u_{r+1}$ be the vertices of a clique of order $r+1$ in C^{n-1}_{nr+k}. Two vertices of $\overline{C^{n-1}_{nr+k}}$ are adjacent if the difference in their suffixes is $\geq$ n. Hence $\overline{C^{n-1}_{nr+k}}$ contains at least

$$(r+1)(n-1) + (r+1) = (r+1)n \tag{5.3.15}$$

vertices. This is not true. So

$$\alpha(C^{n-1}_{nr+k}) = r \tag{5.3.16}$$

by (5.3.12). As a result S of (5.3.11) is a MISS.

$$\alpha(C^{n-1}_{nr+k} - v_i\, v_{i+1}) > r \tag{5.3.17}$$

because in $C^{n-1}_{nr+k} - v_i\, v_{i+1}$, v_i is not adjacent to any of the vertices of S. This means that

$$v_i\, v_{i+1} \text{ is a critical edge.} \tag{5.3.18}$$

The edges $v_i\, v_{i+1}$ $(1 \leq i \leq nr+k-1)$ form a connected spanning subgraph of C^{n-1}_{nr+k}. Hence the graph is indecomposable by (5.1.1). //

In Theorem 4.2.4 we have seen that C^{n-1}_{nr} $(n, r \geq 2)$ is B-point critical. Theorem 5.3.2 states that C^{n-1}_{nr+k}, $(n, r \geq 2)$, $1 \leq k \leq n-1$ is indecomposable. Combining these two results we

observe that for $t \geq 2n$

$$C_t^{n-1} \text{ is B-point critical} \tag{5.3.19}$$

if t is a multiple of n and

$$C_t^{n-1} \text{ is indecomposable otherwise.} \tag{5.3.20}$$

Next we shall see under what conditions the complement of C_{nr+k}^{n-1} is indecomposable. We require the following two results.

<u>LEMMA 5.3.1</u> For $r \leq [\frac{k}{2}] - 1$, $\omega(C_k^r) = r+1$.

<u>Proof</u> : Suppose that $v_1, v_2, v_3, \ldots, v_k, v_1$ is the cycle C_k. Any vertex v_i $(1 \leq i \leq k)$ in C_k^r is adjacent to vertices of

$$S = \{v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{i+r}\} \tag{5.3.21}$$

and

$$S' = \{v_{i+k-1}, v_{i+k-2}, v_{i+k-3}, \ldots, v_{i+k-r}\} \tag{5.3.22}$$

$$< S \cup \{v_i\} > \text{ is a clique of order } r+1. \tag{5.3.23}$$

So $$\omega(C_k^r) \geq r+1 . \tag{5.3.24}$$

In $\overline{C_k^r}$, the vertex v_i is adjacent to all the vertices other than that of S and S'. As a result, all the independent sets of $\overline{C_k^r}$ containing v_i have vertices from S and S' only. $v_{i+1}, v_{i+k-r}, v_{i+2}, v_{i+k-(r-1)}, v_{i+3}, v_{i+k-(r-2)}, \ldots, v_{i+r-1}, v_{i+k-2}, v_{i+r}, v_{i+k-1}$ is a path with 2r vertices in C_k^r. Therefore we can select only r vertices for an independent set containing v_i. $S \cup \{v_i\}$ is an independent set of order r+1 in $\overline{C_k^r}$ by (5.3.23). Hence

$$\alpha(\overline{C_k^r}) = r+1 . \tag{5.3.25}$$

This implies that

$$\omega(C_k^r) = r+1. \ //$$

<u>LEMMA 5.3.2</u> If n and r are positive integers ≥ 2 and $k (1 \leq k \leq n-1)$ is an integer relatively prime to n, then the remainders obtained when $1.n, 2.n, 3.n, \ldots, (nr+k-1)n$ are divided by $nr+k$ are all distinct.

<u>Proof</u> : Suppose that $i.n$ and $j.n$ $(1 \leq i, j \leq nr+k-1)$ leave the same remainder R when divided by $nr+k$. Let

$$i.n = \lambda(nr+k)+R \tag{5.3.26}$$

$$j.n = \mu(nr+k)+R \tag{5.3.27}$$

where $\lambda, \mu \leq n$. Then

$$(i-j)n = (\lambda-\mu)(nr+k). \tag{5.3.28}$$

The numbers n and $nr+k$ are relatively prime. From equation (5.3.28) we see that a multiple of n is equal to a multiple of $nr+k$. Hence

$$i-j \text{ is a multiple of } nr+k\ . \tag{5.3.29}$$

Since i and j are separately less than or equal to $nr+k-1$, (5.3.29) is not true. Therefore, (5.3.26) and (5.3.27) lead to a contradiction. //

Lemma 5.3.2 helps us to know whether there exists a connected spanning subgraph of $C_{nr+k}^{\overline{n-1}}$. In this graph two vertices are adjacent if the difference in their suffixes is $\geq n$, provided that we label the vertices of $C_{nr+k}^{\overline{n-1}}$ consecutively from

v_1 to v_{nr+k}. Thus any vertex v_i is adjacent to v_{i+n}. This in turn is adjacent to v_{i+2n} and so on. By the Lemma 5.3.2, in the sequence $v_i, v_{i+n}, v_{i+2n}, v_{i+3n}, \ldots, v_{i+(r-1)n+k}, v_i$ all the vertices of $\overline{C^{n-1}_{nr+k}}$ are present and no vertex other than v_i appears more than once. Hence the above sequence determines a spanning cycle of $\overline{C^{n-1}_{nr+k}}$. This information helps us to prove the next theorem.

THEOREM 5.3.3 $\overline{C^{n-1}_{nr+k}}$, $n, r \geq 2$, n and k $(1 \leq k \leq n-1)$ relatively prime are indecomposable.

Proof : From Lemma 5.3.1 we know that

$$\omega(C^{n-1}_{nr+k}) = n \tag{5.3.30}$$

Hence

$$\alpha(\overline{C^{n-1}_{nr+k}}) = n . \tag{5.3.31}$$

So

$$S = \{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+n-1}\} \tag{5.3.32}$$

is a MISS in $\overline{C^{n-1}_{nr+k}}$. The vertex v_{i+n} is adjacent to no vertex of S other than v_i. Therefore

$$\alpha(\overline{C^{n-1}_{nr+k}} - v_i\, v_{i+n}) = n+1 . \tag{5.3.33}$$

Thus $v_i\, v_{i+n}$ is a critical edge. Edges (v_i, v_{i+n}), (v_{i+n}, v_{i+2n}), (v_{i+2n}, v_{i+3n}), etc. form a connected spanning subgraph of $\overline{C^{n-1}_{nr+k}}$ by Lemma 5.3.2. This means that $\overline{C^{n-1}_{nr+k}}$ is indecomposable by (5.1.1). //

Next we supply a few results which will help us to establish that line graphs of the powers of odd cycles are line critical and hence indecomposable.

LEMMA 5.3.3 If (v_i, v_j) and (v_i, v_t) are any two adjacent edges of C^r_{2n+1}, $r \geq 1$, then $C^r_{2n+1} - \{v_i, v_j, v_t\}$ has a perfect matching.

Proof : First of all we observe that P_{2n} has a perfect matching. It is enough to prove that all the vertices of $C^r_{2n+1} - \{v_i, v_j, v_t\}$ lie on P_{2n-2}. In the case of C_{2n+1}, when three consecutive vertices are removed we get P_{2n-2}. In order to show that all the vertices of $C^r_{2n+1} - \{v_i, v_j, v_t\}$ $(r \geq 2)$ lie on P_{2n-2} we consider the following cases.

Case 1 v_i, v_j, v_t are consecutive vertices of C^r_{2n+1}. Let $j = i+1$ and $t = j+1$. In this case the vertices $v_{t+1}, v_{t+2}, v_{t+3}, \ldots, v_{t+2n-2}$ lie on a P_{2n-2}.

Case 2 Two of the vertices of $\{v_i, v_j, v_t\}$, say v_j and v_t are consecutive.

In this case all the edges of the cycle C_{2n+1} other than (v_{j-1}, v_j), (v_j, v_t), (v_t, v_{t+1}), (v_{i-1}, v_i), (v_i, v_{i+1}) and the chord (v_{i-1}, v_{i+1}) form a P_{2n-2}.

Case 3 All the vertices of $\{v_i, v_j, v_t\}$ are separated by other vertices of the cycle.

In the present case taking all the edges of the cycle C_{2n+1} which are not incident with v_i, v_j and v_t and the

chords (v_{i-1}, v_{i+1}), (v_{j-1}, v_{j+1}) and (v_{t-1}, v_{t+1}) we get a cycle C_{2n-2} which has a perfect matching. //

COROLLARY 5.3.2 For any two adjacent edges (v_i, v_j) and (v_i, v_t) of C^r_{2n+1} there exists a set S of n-1 independent edges in C^r_{2n+1} such that each of the sets $(v_i, v_j) \cup S$ and $(v_i, v_t) \cup S$ is independent.

Proof : From Lemma 5.3.3 we can see that $C^r_{2n+1} - \{v_i, v_j, v_t\}$ has a perfect matching S having n-1 edges. The vertices v_i, v_j and v_t do not lie on any of the edges of S. Therefore the edges (v_i, v_j) and (v_i, v_t) have no points in common with any end point of an edge in S. Hence the result follows. //

THEOREM 5.3.4 $L(C^r_{2n+1})$, $(1 \leq r \leq n)$ is line critical.

Proof : From Lemma 3.3.4,

$$\beta(C^r_{2n+1}) = n.$$

Hence $$\alpha(L(C^r_{2n+1})) = n . \tag{5.3.34}$$

Each edge in the line graph of a graph G corresponds to a pair of adjacent edges of G. Hence the removal of an edge in L(G) is equivalent to breaking the adjacency of two adjacent edges of G. If e is any edge of $L(C^r_{2n+1})$, then

$$\alpha(L(C^r_{2n+1})-e) = n+1 . \tag{5.3.35}$$

by Corollary 5.3.2, since in the present case the edges (v_i, v_j) and (v_i, v_t) form an independent set with n-1 edges of S. Therefore $L(C^r_{2n+1})$ $(1 \leq r \leq n)$ is line critical and hence indecomposable by (5.1.2). //

<u>COROLLARY 5.3.3</u> $L(K_{2n+1})$ is line critical.

<u>COROLLARY 5.3.4</u> $L(K_{2n})$ is decomposable.

<u>Proof</u> : We have

$$\alpha(L(K_{2n})) = n \tag{5.3.36}$$

since K_{2n} has n independent edges. Also

$$\alpha(L(K_{2n-1})) = n-1 \tag{5.3.37}$$

Let V_1 represent the 2n-1 vertices of $L(K_{2n})$ corresponding to the 2n-1 edges incident with a vertex of K_{2n}. Let V_2 stand for the remaining vertices of $L(K_{2n})$. Then the induced subgraphs of $L(K_{2n})$ on V_1 and V_2 are K_{2n-1} and $L(K_{2n-1})$ respectively. Now,

$$\alpha(L(K_{2n})) = \alpha(L(K_{2n-1})) + \alpha(K_{2n-1}) \tag{5.3.38}$$

using (5.3.36) and (5.3.37) since $\alpha(K_{2n-1}) = 1$. V_1 and V_2 partition the vertices of $L(K_{2n})$. As a result,

$$[L(K_{2n-1}), K_{2n-1}]$$

is a decomposition of $L(K_{2n})$ by (5.3.38). //

<u>COROLLARY 5.3.5</u> $T(K_{2n})$ is line critical.

Proof follows by observing that $T(K_p)$ is isomorphic to $L(K_{p+1})$ [5] . //

<u>COROLLARY 5.3.6</u> $T(K_{2n+1})$ is decomposable.

This follows from the Corollary 5.3.4 since $L(K_{2n+2})$ is decomposable and $T(K_{2n+1})$ is isomorphic to $L(K_{2n+2})$. //

The next theorem gives a property of indecomposable graphs.

THEOREM 5.3.5 For any clique Q of an indecomposable graph G, $\alpha(G-Q) = \alpha(G)$.

Proof : Suppose that G is an indecomposable graph with

$$\alpha(G-Q) < \alpha(G) . \tag{5.3.39}$$

Since Q is a clique, the minimum value of $\alpha(G-Q)$ is $\alpha(G)-1$. Hence

$$\alpha(G-Q) + \alpha(Q) = \alpha(G) , \tag{5.3.40}$$

contradicting the indecomposability of G. //

The converse of this result is not true. Petersen graph is one example. The graph in Figure 4.3.1 also has the property $\alpha(G-Q) = \alpha(G)$ for every clique of G. But this graph is decomposable since $[G_1, G_2]$ is a decomposition by (5.2.6).

CHAPTER 6

ON THE STRONG PERFECT GRAPH CONJECTURE

CHAPTER 6

On The Strong Perfect Graph Conjecture

6.1 Introduction

The strong perfect graph conjecture (SPGC) states that for a graph G the following conditions are equivalent.

1) G is α-perfect

2) G is χ-perfect

3) G and $\overline{G}$ do not contain an induced subgraph isomorphic to C_{2k+1}, $k \geq 2$. The equivalence of the first two conditions is established by Lovasz [71]. He supplied a good characterization of perfect graphs also [72]. As a result, the SPGC may be stated in the following simple form: A graph G is perfect if and only if it does not contain C_{2k+1} or $\overline{C_{2k+1}}$, $k \geq 2$ as an induced subgraph of G. Tucker ([122], [124]) proved that the SPGC is true for planar graphs and circular arc graphs. Parthasarathy and Ravindra ([92], [93]) established this conjecture for $K_{1,3}$-free graphs and (K_4-e)- free graphs. Trotter [118] showed that webs satisfy the SPGC. In Chapter 1 we saw that this conjecture holds good for $\overline{K_{1,3}}$ - free graphs.

In the present chapter we prove that each of the graphs C_{2n}^k and C_{2n+1}^k for all k, $2 \leq k \leq n-2$ contains either an odd cycle of length ≥ 5 or its complement as an induced subgraph. We establish that all the remaining powers of the above cycles other than C_{2n+1} and C_{2n+1}^{n-1} are perfect. Combining these facts

we show that the SPGC is true for powers of cycles. We pose four conjectures and prove that each of them is equivalent to the SPGC.

6.2 Validity of Berge's Conjecture for Powers of Cycles

By [48], C_{2n+1}, $n \geq 2$ is an imperfect graph. Also $\overline{C_{2n+1}}$ is an imperfect graph since the complement of a perfect graph is perfect [71]. Hence from the definition of a perfect graph we can see that if a graph G contains C_{2n+1}, $n \geq 2$ or its complement as an induced subgraph, then G is imperfect. Using this we prove that each of the graphs C^k_{2n} and C^k_{2n+1}, $2 \leq k \leq n-2$ is imperfect. We consider the cases $k = n-1$ and $k = n$ separately. We need the following results.

LEMMA 6.2.1 Given an integer k, $2 \leq k \leq [\frac{I}{2}] - 2$, any integer I may be expressed in the form $I = p(k+1)+r$ uniquely where p and r are integers and $o \leq r \leq k$.

Proof : Consider the following equations corresponding to the k+1 values of r.

$$I = p(k+1) + 0$$
$$I = p(k+1) + 1$$
$$I = p(k+1) + 2$$
$$\cdots\cdots\cdots$$
$$I = p(k+1) + k .$$

Equivalently

$$p(k+1) = I$$
$$p(k+1) = I-1$$
$$p(k+1) = I-2$$
$$\cdots\cdots\cdots$$
$$p(k+1) = I-k$$

The right hand sides of these equations are k+1 consecutive integers. Out of n consecutive integers one and only one is a multiple of n. Hence the resolution as supplied in the Lemma holds good and it is unique. //

<u>Remark 6.2.1</u> Lemma 6.2.1 is true even if the bounds of k are not as tight as given here. We choose these bounds so that it is readily applicable in Theorem 6.2.1.

<u>LEMMA 6.2.2</u> If $I = p(k+1)+r$, $2 \leq k \leq [\frac{I}{2}] - 2$, $o \leq r \leq k$, then $p \geq 2$.

<u>Proof</u> : We shall prove that

$$p = o \tag{6.2.1}$$

and
$$p = 1 . \tag{6.2.2}$$

are inadmissible values of p. If (6.2.1) is true, then

$$I = r. \tag{6.2.3}$$

The maximum value of r is k.

$$k \leq [\frac{I}{2}] - 2 \tag{6.2.4}$$

by hypothesis. Hence

$$r \leq [\tfrac{I}{2}] - 2. \tag{6.2.5}$$

Now (6.2.5) contradicts (6.2.3). If (6.2.2) holds good, then

$$I = k+1-r . \tag{6.2.6}$$

$$k+1+r \leq 2k+1 \tag{6.2.7}$$

since $\quad r \leq k.$

$$2k+1 \leq 2([\tfrac{I}{2}] - 2)+1 \tag{6.2.8}$$

by hypothesis. Using (6.2.8) and (6.2.7) in (6.2.6), we get

$$I \leq I-3.$$

This also is not possible. Hence

$$p \geq 2. //$$

<u>THEOREM 6.2.1</u> C^k_{2n}, $2 \leq k \leq n-2$ has C_{2s+1} or $\overline{C_{2s+1}}$, $s \geq 2$ as an induced subgraph.

<u>Proof</u> : Let p be an integer such that

$$2n = p(k+1)+r, \quad 0 \leq r \leq k. \tag{6.2.9}$$

Then $\quad p \geq 2 \quad$ (6.2.10)

by Lemma 6.2.2.

<u>Case 1</u> $\quad p = 2.$

We have

$$2n = 2(k+1)+r. \tag{6.2.11}$$

If $\quad r = 0 \quad$ (6.2.12)

$$2n = 2(k+1). \tag{6.2.13}$$

$$k \leq n-2 \tag{6.2.14}$$

by hypothesis. Therefore

$$2(k+1) \leq 2(n-1). \tag{6.2.15}$$

Now (6.2.15) contradicts (6.2.13). Hence

$$r > o . \tag{6.2.16}$$

From (6.2.11) we get that r is an even integer. So

$$r \geq 2. \tag{6.2.17}$$

Let $v_1, v_2, v_3, \ldots, v_{2n}, v_1$ be the cycle C_{2n}. Each vertex v_i in C^k_{2n} is adjacent to r+1 vertices namely $v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, \ldots, v_{i+k+r+1}$ (suffix modulo 2n) because of (6.2.11). Consider the sequence of vertices $v_1, v_{k+2}, v_{2k+3}, v_{3k+4}, v_{4k+5}, \ldots, v_{tk+t+1}, \ldots$. Each vertex of this sequence is adjacent to the vertex preceding it and the vertex succeeding it respectively. Using the modulo property the above sequence may be written in the following form :

$v_1, v_{k+2}, v_{2k+3}, v_{k+2-r}, v_{2k+3-r}, v_{k+2-2r}, v_{2k+3-2r}, \ldots, v_{k+2-tr}, v_{2k+3-tr}, \ldots$. The vertex $v_{2k+3-tr}$ is adjacent to v_1 if and only if

$$k+2 < 2k+3-tr \leq k+r+2 . \tag{6.2.18}$$

This is the same as

$$tr-1 < k \leq (t+1)r-1 . \tag{6.2.19}$$

If $$t = 0 \tag{6.2.20}$$

we get $$k \leq r-1 \ . \tag{6.2.21}$$

This goes against the hypothesis. Hence

$$t \geq 1 \ . \tag{6.2.22}$$

The restriction (6.2.19) on the integer t may be expressed in terms of n and r by substituting for k from (6.2.11). That is

$$tr-1 < n-1 - \frac{r}{2} \leq (t+1)r-1$$

which may be simplified as

$$\frac{n}{r} - \frac{3}{2} \leq t < \frac{n}{r} - \frac{1}{2} \ . \tag{6.2.23}$$

All the vertices in the sequence

$$S = \{v_1, v_{k+2}, v_{2k+3}, v_{k+2-r}, v_{2k+3-r}, v_{k+2-2r}, \\ v_{2k+3-2r}, \ldots, v_{k+2-tr}, v_{2k+3-tr}\} \tag{6.2.24}$$

are distinct since the numbers 1, $k+2$ and $2k+3$ are all $< 2n$, $r > o$ and t satisfies (6.2.23).

Next we shall prove that $\langle S \rangle$ of $\overline{C^k_{2n}}$ is C_{2t+3} provided that t satisfies (6.2.19). In order to establish this we need prove only that each vertex of S is not adjacent to more than two of its vertices. We observe that

$$1 < k+2-tr \leq k \tag{6.2.25}$$

and $$k+3 \leq 2k+3-tr \leq 2k+1 \tag{6.2.26}$$

since $k > tr-1$ and the minimum value of tr is 2. This implies that the vertices of

$$S_1 = \{v_{k+2-r}, v_{k+2-2r}, \ldots, v_{k+2-tr}\} \tag{6.2.27}$$

may be any one of the vertices $v_2, v_3, v_4, \ldots, v_k$ and the vertices of

$$S_2 = \{v_{2k+3-r}, v_{2k+3-2r}, \ldots, v_{2k+3-tr}\} \tag{6.2.28}$$

belong to the set $\{v_{k+3}, v_{k+4}, \ldots, v_{2k+1}\}$. From this it is clear that no two vertices of S_1 are adjacent. Similarly for S_2. Now we shall prove that the vertex v_{k+2-jr} is adjacent only to $v_{2k+3-(j-1)r}$ and $v_{2k+3-jr}$. There are $r+1$ consecutive integers starting with $2k+3-(j-1)r$ and ending in $2k+3-jr$. Since v_{k+2-jr} is adjacent to $v_{2k+3-jr}$ and $v_{2k+3-(j-1)r}$ it cannot be adjacent to any vertex beyond the above range. As a result, v_{k+2-jr} is not adjacent to any vertex of S_2 other than $v_{2k+3-(j-1)r}$ and $v_{2k+3-jr}$. By an exactly similar argument we can prove that $v_{2k+3-jr}$ is adjacent only to v_{k+2-jr} and $v_{k+2-(j+1)r}$. Because of (6.2.25), v_1 is not adjacent to any vertex of S_1. The vertex v_1 is not adjacent to any vertex $v_{2k+3-ir}$ $(1 \leq i \leq t-1)$, because in that case

$$tr-1 < k \leq (t+1)r-1$$

and

$$ir-1 < k \leq (i+1)r-1 \ .$$

This is not possible since $i \leq t-1$. The vertex v_{2k+3} is adjacent to v_{k+2-r} and v_{k+2} and all vertices which fall between them. Hence v_{2k+3} is not adjacent to the vertices v_{k+2-2r},

$v_{k+2-3r}, \ldots, v_{k+2-tr}$. It is not adjacent to v_1 also. The vertex v_{k+2} is adjacent to the r+1 vertices starting from v_{2k+3} and ending in v_1. Hence it is not adjacent to the vertices $v_{2k+3-r}, v_{2k+3-2r}, \ldots, v_{2k+3-tr}$. It is also not adjacent to the vertices of S_1. Combining all these we get that the subgraph of $\overline{C^k_{2n}}$ induced on S is C_{2t+3}. So C^k_{2n} has $\overline{C_{2t+3}}$ as an induced subgraph.

Case 2 $\quad p \geq 3$.

For convenience we may divide this into three simpler cases.

Case 2a) $\quad o \leq r \leq k-2$.

We have

$$2n = pk+p+r \quad (6.2.29)$$

Let $v_1, v_2, v_3, \ldots, v_{2n}, v_1$ be the cycle C_{2n}. Any vertex v_i in C^k_{2n} is adjacent to

$$S_3 = \{v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{i+k}\} \quad (6.2.30)$$

and

$$S_4 = \{v_{i+2n-1}, v_{i+2n-2}, \ldots, v_{i+2n-k}\} \quad (6.2.31)$$

(suffix modulo 2n). Consider the set

$$S_5 = \{v_1, v_{k+1}, v_{k+2}, v_{2k+2}, v_{2k+3}, v_{3k+3}, \ldots, v_{(p-2)k+p-1}, v_{(p-1)k+p-1}, v_{pk+p-(k-r-1)}\} \quad (6.2.32)$$

of vertices of C^k_{2n}. All the vertices of S_5 are distinct since

each of the suffixes is $< 2n$, and

$$pk+p-(k-r-1) > (p-1)k+p-1 \quad . \tag{6.2.33}$$

Each of the vertices $v_{k+1}, v_{k+2}, v_{2k+2}, v_{2k+3}, v_{3k+3}, \ldots, v_{(p-1)k+p-1}$ is adjacent only to the vertex preceding it and the vertex succeeding it respectively beeause of the adjacency defined by (6.2.30) and (6.2.31). The vertex v_1 is adjacent to v_{k+1} and $v_{pk+p-(k-r-1)}$ since

$$pk+p-(k-r-1)+k = 1 \pmod{2n} \quad . \tag{6.2.34}$$

The vertex $v_{(p-1)k+p-1}$ is adjacent to $v_{pk+p-(k-r-1)}$ if

$$pk+p-(k-r-1) - [(p-1)k+p-1] \leq k \quad . \tag{6.2.35}$$

That is, if $\qquad r+2 \leq k$. $\hfill$ (6.2.36)

This is true by the hypothesis.

Hence the subgraph of C_{2n}^{k} induced on S_5 is C_{2p-1}.

Case 2b) $\quad r = k-1$.

In this case

$$2n = (p+1)k+p-1 \quad . \tag{6.2.37}$$

Consider the vertices $v_1, v_{k+1}, v_{2k+1}, v_{2k+2}, v_{3k+2}, v_{3k+3}, \ldots, v_{(p-2)k+p-2}, v_{(p-1)k+p-2}, v_{(p-1)k+p}, v_{pk+p}, v_1$. As in the previous case, we can see that each vertex is adjacent to the vertex preceding it and the vertex suceeding it respectively. The vertex v_{pk+p} is adjacent to v_1 since

$$pk+p+k = 1 \bmod (2n) \tag{6.2.38}$$

by (6.2.37). Therefore the above vertices form C_{2p-1} in C_{2n}^{k}.

Case 2c) $r = k$.

We have

$$2n = pk+p+k . \qquad (6.2.39)$$

In this case $v_1, v_{k+1}, v_{k+2}, v_{2k+2}, v_{2k+3}, v_{3k+3}, \ldots,$ $v_{(p-1)k+p}, v_{pk+p}, v_{pk+p+1}, v_1$ is C_{2p+1} since

$$pk+p+1+k = 1 \pmod{2n}. \text{ //}$$

THEOREM 6.2.2 C_{2n+1}^{k}, $2 \leq k \leq n-2$ has C_{2s+1} or $\overline{C_{2s+1}}$, $s \geq 2$ as an induced subgraph.

Proof : We observe that all the cases discussed in Theorem 6.2.1 are applicable here also. But in this case (6.2.17) is to be replaced by

$$r \geq 1 . \qquad (6.2.40)$$

Accordingly (6.2.25) and (6.2.26) are to be modified as

$$1 < k+2-tr \leq k+1 \qquad (6.2.41)$$

$$k+3 \leq 2k+3-tr \leq 2k+2 . \qquad (6.2.42)$$

The induced odd cycles of length ≥ 5 or its complement obtained in the respective cases of Theorem 6.2.1 are valid as such here also. Hence the result follows. //

Now we consider the remaining powers of C_{2n} and C_{2n+1} in order to prove Berge's conjecture for powers of cycles.

LEMMA 6.2.3 C_{2n} is a perfect graph.

Proof follows since C_{2n} is bipartite.

LEMMA 6.2.4 C_{2n}^{n-1} is a perfect graph.

Proof : In C_{2n}^{n-1} each vertex v_i is adjacent to 2n-2 vertices. Hence the vertex v_i is not adjacent to exactly one vertex. Therefore in $\overline{C_{2n}^{n-1}}$, v_i is adjacent to exactly one vertex. As a result, $\overline{C_{2n}^{n-1}}$ is a collection of n copies of K_2. For a complete graph

$$\chi = \omega \tag{6.2.43}$$

by [4]. Every induced subgraph of a complete graph is also complete. Hence

a complete graph is perfect. (6.2.44)

So, $\overline{C_{2n}^{n-1}}$ and C_{2n}^{n-1} are perfect graphs [71].

LEMMA 6.2.5 (V. Chvatal [27]) C_{2n+1}^{n-1} is isomorphic to $\overline{C_{2n+1}}$.

LEMMA 6.2.6 C_{2n}^{n} and C_{2n+1}^{n} are perfect graphs.

Proof : The result follows since C_{2n}^{n} and C_{2n+1}^{n} are complete graphs.

THEOREM 6.2.3 Berge's conjecture is true for powers of cycles.

Proof : From Lemmas 6.2.3, 6.2.4 and 6.2.6, we see that C_{2n}^{k} is perfect for k=1, n-1 and n. All the remaining powers of C_{2n}

have at least one odd cycle of length $\geq$ 5 or its complement as an induced subgraph by Theorem 6.2.1. Hence Berge's conjecture is valid for powers of even cycles.

C_{2n+1}^{k} reduces to an odd cycle when $k = 1$. By Lemma 6.2.5 C_{2n+1}^{n-1} is $\overline{C_{2n+1}}$. C_{2n+1}^{n} is a perfect graph by Lemma 6.2.6. From Theorem 6.2.2, C_{2n+1}^{k}, $2 \leq k \leq n-2$, has C_{2s+1} or $\overline{C_{2s+1}}$, $s \geq 2$ as an induced subgraph. Hence the SPGC holds good for powers of odd cycles also. //

6.3 Equivalent forms of Berge's conjecture

The SPGC is equivalent to the statement that a critical graph is C_{2n+1} or $\overline{C_{2n+1}}$, $n \geq 2$. We give four equivalent forms and establish that each of them is equivalent to the SPGC so that the solution of any one of them will settle the Berge's conjecture.

Conjecture 6.3.1 For a critical graph G,

$$\omega + \alpha = \frac{p+3}{2} .$$

For all critical graphs

$$\omega \alpha = p-1 \tag{6.3.1}$$

by (2.1.16), If, for a critical graph

$$\omega + \alpha = \frac{p+3}{2} \tag{6.3.2}$$

then combining (6.3.1) and (6.3.2) we get

$$\omega = 2 \tag{6.3.3}$$

or

$$\omega = \frac{p-1}{2} . \tag{6.3.4}$$

If (6.3.3) is true, then

$$G = C_{2n+1}, \ n \geq 2 \tag{6.3.5}$$

by (2.1.17). If (6.3.4) holds good, then

$$\alpha = 2 \tag{6.3.6}$$

by (6.3.1). So

$$G = \overline{C_{2n+1}}, \ n \geq 2 \tag{6.3.7}$$

by (2.1.18).

<u>Conjecture 6.3.2</u> For a critical graph G,

$$\alpha(H_1(v)) = 2 \ .$$

If $\alpha(H_1(v)) = 2$ (6.3.8)

the graph G is $K_{1,3}$ -free and

$$d(v) = 2\omega - 2 \tag{6.3.9}$$

by Corollary 2.3.2. Since $\overline{G}$ is also critical

$$\alpha(H_1(\overline{G},v)) = 2 \ . \tag{6.3.10}$$

So $d(\overline{G},v) = 2\omega(\overline{G}) - 2$ (6.3.11)

This implies that

$$|H_2(G,v)| = 2\alpha - 2 \ . \tag{6.3.12}$$

Hence $d(G,v) = p-1-(2\alpha-2)$ (6.3.13)

using (2.2.45). Now equating the two values of d(v) from (6.3.9) and (6.3.13) we get

$$\omega + \alpha = \frac{p+3}{2} \ .$$

This conjecture may also be put in the following form : For all critical graphs G, $\omega(H_2(v)) = 2$.

<u>Conjecture 6.3.3</u> For all critical graphs G,

$$|H_1(v)| \cdot |H_2(v)| = 2(p-3) .$$

We have for all graphs

$$|H_1(v)| + |H_2(v)| = p-1 . \tag{6.3.14}$$

If for critical graphs

$$|H_1(v)| \cdot |H_2(v)| = 2(p-3) \tag{6.3.15}$$

then combining (6.3.14) and (6.3.15) we get

$$|H_1(v)| = 2 \tag{6.3.16}$$

or $$|H_1(v)| = p-3 . \tag{6.3.17}$$

If (6.3.16) is true, then

$$2 \geq 2\omega - 2 . \tag{6.3.18}$$

But $$\omega \geq 2 \tag{6.3.19}$$

by (2.1.9). From (6.3.18) and (6.3.19)

$$\omega = 2 .$$

So $$G = C_{2n+1}, \; n \geq 2$$

by (2.1.17). If (6.3.17) is true, then

$$p-3 \leq p+1-2\alpha \tag{6.3.20}$$

But $$\alpha \geq 2 \tag{6.3.21}$$

by (2.1.1). Combining (6.3.20) and (6.3.21)

$$\alpha = 2 .$$

Hence $$G = \overline{C_{2n+1}}, \quad n \geq 2$$

by (2.1.18).

<u>Conjecture 6.3.4</u> The number of edges in a critical graph G is $p\omega + 3 - 2(\omega + \alpha)$.

Suppose that the number of edges q of a critical graph is as given. $\bar{G}$ is also a critical graph. Therefore

$$\frac{p(p-1)}{2} - q = p\omega(\bar{G}) + 3 - 2[\omega(\bar{G}) + \alpha(\bar{G})] \qquad (6.3.22)$$

$$= p\alpha + 3 - 2(\alpha + \omega) ,$$

which is the same as

$$\frac{p(p-1)}{2} - p\omega - 3 + 2(\omega + \alpha) = p\alpha + 3 - 2(\omega+\alpha) .$$

This leads to

$$p^2 - p[2(\omega+\alpha) + 1] + 8(\omega+\alpha) - 12 = 0. \qquad (6.3.23)$$

This equation gives

$$p = 4 \qquad (6.3.24)$$

or $$p = 2(\omega+\alpha) - 3 . \qquad (6.3.25)$$

The value given in (6.3.24) is inadmissible as there is no critical graph on 4 vertices by Theorem 2.2.4. Hence

$$\omega + \alpha = \frac{p+3}{2} .$$

It seems that the solution of Berge's strong perfect graph conjecture is extremely difficult. Each of the conjectures given here suggests just one more additional restriction on critical graphs. Therefore establishing any of these restrictions for critical graphs is easier than the solution of the SPGC.

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